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AUTHOR(S):

Lê, Dũng Tráng; Urabe, T.

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Lê Dũng Tráng
Notes prepared by
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Preface

In 1980 Fall, Professors Lê Dũng Tráng of Paris VII and Fluvio Lazzeri of Pisa stayed at the Research Institute for Mathematical Sciences, Kyoto University for several months. At that occasion a big group of active researchers in the theory of singularities, including both resident and visiting members of the Institute, carried out a series of vigorous seminars of approximately 60 sessions, which I hope were extremely stimulating to all participants.

The present volume consists of the lectures given by Professor Lê Dũng Tráng in these seminars. Mr. Tohsuke Urabe took the notes and prepared the manuscript. The Department of Mathematics, Kyoto University has given the chance of publishing these notes in the present form which, I believe, makes them easily accessible to a large number of mathematicians. I would like to express, personally and on behalf of the Research Institute for Mathematical Sciences, deep thanks to Professor Lê Dũng Tráng, to Mr. T. Urabe, to the Department of Mathematics and to all the participants of the seminars. Thanks are also due to the secretaries of the RIMS for their splendid typing work.

There is a plan to publish the notes of other lectures of the seminars. Many of them will be written in Japanese and will appear in other places.

Shigeo Nakano

May 1981

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C O N T E N T S

Geometry of Tangents on Singular Spaces and Chern Classes

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Lê Dũng Tráng

(Notes prepared by T. Urabe)

Chapter I. Geometry of Tangents

Let $(X,0) \subset (\mathbb{C}^N,0)$ be a germ of analytic space. For a smooth point $x \in X$, we can define the tangent space T_x of X at x. In this chapter we study the behaviour of T_x when x tends to the origin 0, which may be singular.

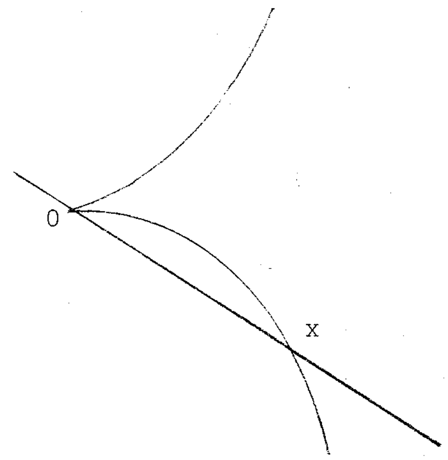
§1. Nash Modification

Let $(X,0) \subset (\mathbb{C}^N,0)$ be a germ of reduced analytic space. We consider a representant X of $(X,0)$ which is the space of common zeros of a finite number of analytic functions defined on an open subset $U \subset \mathbb{C}^N$. We assume that $(X,0)$ has pure dimension d.

There is a natural map

$$\mathbb{C}^N-(0) \longrightarrow \mathbb{P}^{N-1}, \quad (a_1,\cdots,a_N) \longmapsto (a_1:\cdots:a_N).$$

Let λ denote the restriction of this map to $X-(0)$. For $x \in X-(0)$, $\lambda(x)$ is <u>the secant at 0 through x.</u>



secant at 0 through x

Let $X' = \overline{Gr.\lambda} \subset X \times \mathbb{P}^{N-1}$ be the closure of the graph $Gr\ \lambda$ of λ : $Gr\ \lambda = \{(x,\overline{0x}) \in \mathbb{C}^N \times \mathbb{P}^{N-1} \mid x \in X-(0)\}$

<u>Lemma 1.1</u> (Remmert). $X' = \overline{Gr\ \lambda}$ is a reduced analytic subspace.

As for the proof, see H. Whitney [18].

<u>Definition 2</u>. Let $\mathcal{J}$ be the ideal of $(X,0)$ in $(\mathbb{C}^N,0)$. Every element $\varphi \in \mathcal{J}$ can be expressed as

$$\varphi = \sum_{k=m}^{\infty} \varphi_k \qquad \begin{array}{l}\varphi_k \text{ is a homogeneous polynomial of} \\ \text{degree } k. \quad \varphi_m \neq 0.\end{array}$$

$\varphi_m = In_0 \varphi$ is called the <u>initial part</u> of φ. Let I be the ideal in $\mathbb{C}[X_1,\cdots,X_N]$ generated by the set $\{In_0 \varphi \mid \varphi \in \mathcal{J}\}$. The <u>tangent cone</u> $C_{X,0}$ of X at 0 is the analytic variety defined by the homogeneous ideal I.

<u>Remark 1.3.</u>

(1) If $\mathcal{J}$ is principal i.e. $\mathcal{J} = (f)$ for some element f, $I = (In_0 f)$.

(2) If $I = (in_0 f_1, \cdots, in_0 f_r)$, then $\mathcal{J} = (f_1, \cdots, f_r)$.

However, the converse of (2) does not always hold.

<u>Remark 1.4.</u> Let $e : X' \to X$ be the map, induced by the projection $X \times \mathbb{P}^{N-1} \to X$.

(1) The restriction of e to $\mathrm{Gr}\,\lambda \subset X'$ is an isomorphism. $e : \mathrm{Gr}\,\lambda \xrightarrow{\sim} X-(0)$.

(2) $e^{-1}(0) \cong \mathrm{Proj}\ C_{X,0} \subset \mathbb{P}^{N-1}$, where $\mathrm{Proj}\ C_{X,0}$ is the projective variety associated with the cone $C_{X,0} \subset \mathbb{C}^N$.

This $e : X' \to X$ is nothing else but the <u>blowing up</u> of the origin (0). (See R. Hartshorne [3].)

Let Σ be the singular locus of X, $X^0 = X - \Sigma$. We can define the <u>Gauss map</u>

$$\gamma^0 : X^0 = X \setminus \Sigma \longrightarrow G(d,N),$$

where $G(d,N)$ is the Grassmann variety, which parametrizes all linear d spaces passing through the origin in $\mathbb{C}^N$. For $x \in X^0$, $\gamma^0(x)$ is the parallel translation to 0 of the tangent space T_x of X at x.

Let $\tilde{X} = \overline{\mathrm{Gr}\,\gamma^0} \subset X \times G(d,N)$ be the closure of the graph of the map γ^0, and $\nu : \tilde{X} \to X$ be the induced map.

<u>Remark 1.5.</u> If $(X,0) \subset (\mathbb{C}^{d+1},0)$ is a hypersurface, the map $\tilde{X} \to X$ is the blowing up of the Jacobian ideal $J(f) = (\frac{\partial f}{\partial x_0}, \cdots, \frac{\partial f}{\partial x_d})\mathcal{O}_X$, where f is the generator of the ideal $\mathcal{J}$

of X.

This fact also holds for complete intersections $\{f_1 = \cdots = f_k = 0\}$ in $\mathbb{C}^N$. Let $\Delta_1, \cdots, \Delta_r$ $(r = \binom{N}{k})$ be the $k \times k$ minors of the Jacobian matrix

$$\begin{pmatrix} \partial f_1/\partial x_1, & \cdots, & \partial f_1/\partial x_N \\ \vdots & & \vdots \\ \partial f_k/\partial x_1, & \cdots, & \partial f_k/\partial x_N \end{pmatrix}.$$

The map $\gamma' : X^0 \to \mathbb{P}^{r-1}$ defined by $\gamma'(x) = (\Delta_1(x):\cdots:\Delta_r(x))$ coincides with the composition of the Gauss map $\gamma^0 : X^0 \to G(N-k,N)$ with the Plücker imbedding $G(N-k,N) \hookrightarrow \mathbb{P}^{r-1}$. Therefore the closure $\tilde{X} = \overline{Gr\ \gamma^0}$ is isomorphic to the closure of $Gr\ \gamma'$ in $X \times \mathbb{P}^{r-1}$. The map $\overline{Gr\ \gamma^0} \to X$ induced by the projection is by definition the blowing up of the Jacobian ideal.

<u>Remark 1.6</u>. Let X be a union of two germs. $X = X_1 \cup X_2$. Then $\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2 \subset X \times G$. Because we take the closure of smooth parts.

<u>Lemma 1.7.</u> Let $(X,0)$ be a germ of reduced analitic space of pure dimension d.

(1) $(X,0)$ is a union of irreducible components of a reduced complete intersection $(X_1,0)$.

(2) $\tilde{X}$ is contained in $\tilde{X}_1$. Moreover, $\tilde{X} = \overline{\nu_1^{-1}(X-\Sigma_1)}$ where Σ_1 is the singular locus of X_1, and $\nu_1 : \tilde{X}_1 \to X_1$ is the map induced by the projection $X_1 \times G(d,N)$ onto X_1.

(3) The map $\nu : \tilde{X} \to X$ coincides with the blowing up of the ideal $J_1\mathcal{O}_X$, werhe J_1 is the Jacobian ideal of X_1.

Proof It is left for readers as an excise. (Cf. Lê-Teissier [10].)

Corollary 1.8. $\tilde{X}$ is an analytic space. We call this map $\nu: \tilde{X} \to X$, the Nash modification of X.

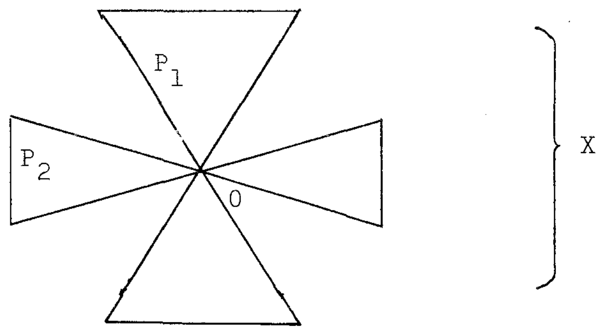
(1) ν is an isomorphism over X^0.

(2) ν is proper.

The tangent bundle TX^0 of the smooth part X^0 is the pull back of the universal bundle over $G(d,N)$ by γ^0. Over $\tilde{X}$, we can define the map $\gamma : \tilde{X} \to G(d,N)$ as the restriction of the projection $X \times G(d,N) \to G(d,N)$. Let $\tilde{T}$ be the pull back of the universal bundle over $G(n,N)$ by γ. We call $\tilde{T}$ the Nash bundle of X.

$$\begin{array}{ccc} \tilde{T} & \hookleftarrow & \nu^*(TX^0) \\ \downarrow & & \downarrow \\ \tilde{X} & \hookleftarrow & \nu^{-1}(X^0) \end{array}$$

Example 1.9 Let $X = P_1 \cup P_2 \subset \mathbb{C}^4$, where P_i $i=1,2$ is a plane in $\mathbb{C}^4$ such that $P_1 \cap P_2 = (0)$.



(1) The Nash modification $\tilde{X}$ is a union of two disjoint plane $\tilde{P}_1$, $\tilde{P}_2$. (Cf. Remark (1.6) above)

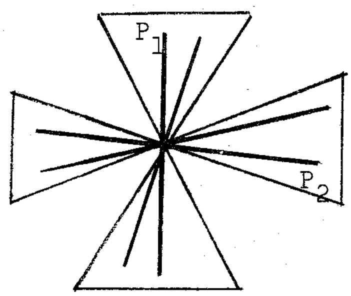
(2) Since the inverse image $\nu^{-1}(0)$ of the origin is two points, it can not be a divisor in $\tilde{X}$ and the map $\nu : \tilde{X} \to X$ is not the blowing-up of any ideal which has its support at the point (0).

(3) Notice that this space X is never a complete intersection. For a sphere $S_\varepsilon = \{(z_1, z_2, z_3, z_\nu) \in \mathbb{C}^a \mid \Sigma |z_i|^2 = \varepsilon\}$ the intersection $X \cap S_\varepsilon$ is a disjoint union of two copies of 3-spheres. If X was a complete intersection $X \cap S_\varepsilon$ should be connected by a local version of Lefschetz theorem.

(4) Let $P_1 = \{z_1 = z_2 = 0\}$, $P_2 = \{z_3 = z_4 = 0\}$, and $X_1 = \{z_1 z_3 = z_2 z_4 = 0\}$. X_1 is a complete intersection containing $X = P_1 \cup P_2$. The Jacobian matrix of X_1 is the following.

$$\begin{pmatrix} z_3, & 0, & z_1, & 0 \\ 0, & z_4, & 0, & z_2 \end{pmatrix}$$

Thus it is easy to see that the Nash modification $\nu: \tilde{X} \to X$ is the blowing-up of the ideal of the union of four coordinate lines.



However, one sees one can choose any set of defining four coordinate lines.

Example 1.10 Let $X = \{f=xz-y^2=0\} \subset \mathbb{C}^3$ be a quadratic cone. For this X, we can see clearly the structure of $\tilde{X}$.

Let $\tilde{X}$ be the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-2)$ on $\mathbb{P}^1$. Then

(1) $\tilde{X} = U_1 \cup U_2$, $U_1 \cong \mathbb{C}^2$, $U_2 \cong \mathbb{C}^2$

(2) $(\zeta,t) \in U_1$ and $(\eta,u) \in U_2$ represent the same point on $\tilde{X}$ if and only if $u = \frac{1}{t}$, $\eta = \zeta t^2$.

(3) There is a map $\nu: \tilde{X} \to X$ such that

$$(\zeta,t) \longmapsto (\zeta,\zeta t,\zeta t^2) = (x,y,z) \quad \text{on } U_1$$

$$(\eta,u) \longmapsto (\eta u^2,\eta u,\eta) = (x,y,z) \quad \text{on } U_2.$$

(4) A map $\gamma : \tilde{X} \to G(2,3) \cong \mathbb{P}^2$ is defined by

$$\gamma(\zeta,t) = (t^2:-2t:1) \qquad \text{on } U_1$$

$$\gamma(\eta,u) = (1:-2u:u^2) \qquad \text{on } U_2.$$

Obviously $(\nu,\gamma): \tilde{X} \to X \times \mathbb{P}^2$ is an embedding, and $(\frac{\partial f}{\partial x}(\nu(p)): \frac{\partial f}{\partial y}(\nu(p)): \frac{\partial f}{\partial z}(\nu(p))) = \gamma(p)$ for a general point $p \in \tilde{X}$. Thus $\nu: \tilde{X} \to X$ is the Nash modification of X.

Now, let $\tilde{T}$ be the Nash bundle over $\tilde{X}$. It should be noted that in this case $\tilde{T} \not\cong T\tilde{X}$. Since the universal bundle over $G(2,3) \cong \mathbb{P}^2$ is isomorphic to $\mathbb{TP}^{2\vee} \otimes \mathcal{O}_{\mathbb{P}^2}(1)$, $\tilde{T} \cong \gamma^*(\mathbb{TP}^{2\vee} \otimes \mathcal{O}_{\mathbb{P}^2}(1))$. Let $E = \nu^{-1}(0)$. E is nothing but the zero-section of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-2)$. It is easy to find the Chern number $c_1(\tilde{T})\cdot E = -2$. On the other hand

$$c_1(T\tilde{X}|_E) = c_1(N_{\tilde{X}/E}) + c_1(TE),$$
$$c_1(N_{\tilde{X}/E})\cdot E = E\cdot E = -2,$$

$c_1(TE)\cdot E = 2$, (since $E \cong \mathbb{P}^1$).

Therefore $c_1(T\tilde{X})\cdot E = 0 \neq c_1(\tilde{T})\cdot E$ and $\tilde{T} \not\cong T\tilde{X}$.

§2. Whitney Lemma

We use the following notation throughout the rest of this article but the last section. Let $(X,0) \subset (\mathbb{C}^N,0)$ be a reduced germ of analytic space of pure dimension d.

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\tilde{e}"] \arrow[d, "\nu'"'] \arrow[dr, "\eta"] & \tilde{X} \arrow[d, "\nu"] \\ X' \arrow[r, "e"'] & X \end{tikzcd}$$

(1) $e : X' \to X$ is the blowing up of the origin.

(2) $\nu : \tilde{X} \to X$ is the Nash modification.

(3) $\tilde{e} : \mathfrak{X} \to \tilde{X}$ is the blowing up of the ideal $m_0 \mathcal{O}_{\tilde{X}}$, where m_0 is the maximal ideal of $\mathcal{O}_{X,0}$.

(4) By the universality of the blowing-up, there is a unique map $\nu' : \mathfrak{X} \to X'$ such that $e\nu' = \nu\tilde{e} = \eta$.

We denote

$$Y' = e^{-1}(0)$$
$$\tilde{Y} = \nu^{-1}(0)$$
$$\mathcal{Y} = \eta^{-1}(0).$$

$\tilde{T}$ is the Nash bundle over $\tilde{X}$ as before. $T = \tilde{e}^*\tilde{T}$ is the pull back of $\tilde{T}$. Let ξ' be the pull back of the canonical line bundle $\mathcal{O}_{\mathbb{P}^{N-1}}(-1)$ over $\mathbb{P}^{N-1}$ by the map $X' \hookrightarrow X \times \mathbb{P}^{N-1} \to \mathbb{P}^{N-1}$, where $X \times \mathbb{P}^{N-1} \to \mathbb{P}^{N-1}$ is the projection. Let $\xi = \nu'^*\xi'$.

$$\begin{tikzcd} \xi' \arrow[d] & & \xi \arrow[d] & T \arrow[dl] & & \tilde{T} \arrow[d] \\ X' & & \mathcal{X} \arrow[ll, "\nu'"'] \arrow[rrr, "\tilde{e}"] & & & \tilde{X} \end{tikzcd}$$

Remark 2.1.

- Y' and $\mathcal{Y}$ are Cartier divisors.
- However $\tilde{Y}$ is not necessarily a divisor.

We denote by $|Z|$ the underlying topological space of the analytic space Z.

Lemma 2.2. (Whitney lemma)

$$\xi|_{|\mathcal{Y}|} \subset T|_{|\mathcal{Y}|}.$$

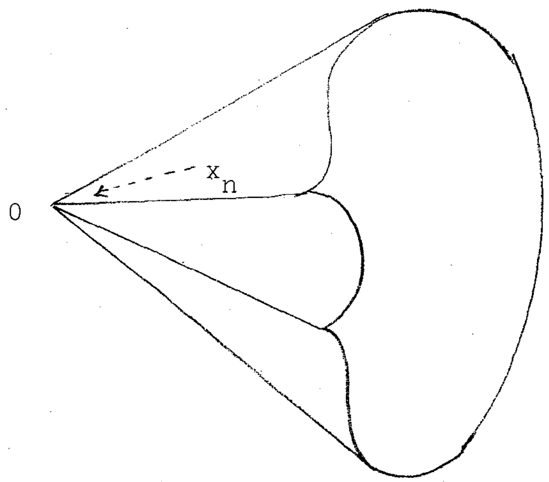
Remark 2.3. The meaning of this lemma is taken in one of the following ways.

(1) By definition, ξ and T are subbundles of the trivial bundle $\mathbb{C}^N_{\mathcal{X}}$. After restricting on $|\mathcal{Y}|$, $\xi|_{|\mathcal{Y}|}$ is included in $T|_{|\mathcal{Y}|}$.

In other words,

(2) (Cf. H. Whitney [18]) For every sequence $x_n \in X^0 = X - \Sigma$, such that

$\mathfrak{I}$ $\lim\limits_{n\to 0} x_n = 0$ (the origin).



② There exists the limit,

$$T = \lim_{n\to\infty} T_{x_n} X.$$

③ There exists the limit

$$\ell = \lim_{n\to\infty} \overline{Ox_n}$$

we have $\ell \subset T$.

(3) $\mathfrak{X}$ can be regarded as a subspace of $X \times \mathbb{P}^{N-1} \times G(n,N)$.

Let

$$\mathcal{H} = \{(\ell, P) \in \mathbb{P}^{N-1} \times G(n,N) \mid \ell \subset P\}.$$

Then, $|\mathcal{Y}| \subset \{0\} \times \mathcal{H}$.

<u>Remark 2.4</u>. In the condition (2), we shall see that for a general limit line ℓ, we have $T = T_\ell |C_{X,0}|$, where we denote by $T_\ell |C_{X,0}|$ the tangent space of the cone $|C_{X,0}|$ at the general point of the generatrix ℓ.

Proof of Whitney lemma. We introduce a continuous function

$$\alpha : X^0 \to \mathbb{R}$$

in order to measure the angle between $\overline{Ox}$ and T_xX.

Let $<v,w>$ be a Hermitian metric of $\mathbb{C}^N$.

Let

$$\alpha(x) = \sup_{\substack{v\in(\overline{Ox})-(0)\\ w\in T_xX-(0)}} \frac{|<v,w>|}{\|v\|\,\|w\|}.$$

It is obvious that:

$$0 \le \alpha(x) \le 1$$

$$\alpha(x) = 1 \iff \overline{Ox} \subset T_xX.$$

Similarly we can define

$$\tilde{\alpha} : \begin{array}{c}\mathfrak{X} \\ \cap \\ X\times\mathbb{P}^{N-1}\times G(n,N)\end{array} \longrightarrow \mathbb{R} .$$

For $(x,\ell,P) \in X \times \mathbb{P}^{N-1} \times G(n,N)$, we set

$$\tilde{\alpha}(x,\ell,P) = \sup_{\substack{v\in\ell-(0)\\ w\in P-(0)}} \frac{|(v,w)|}{\|v\|\|w\|} .$$

Note that for every point $\tilde{x}\in\mathfrak{X}$ such that $\eta(\tilde{x}) \in X^0$, $\tilde{\alpha}(\tilde{x}) = \alpha(\eta(\tilde{x}))$.

What we want to prove now is that:

$$\tilde{\alpha}|_{|\mathcal{Y}|} \equiv 1.$$

It is obvious that the lemma follows from this assertion.

Pick an arbitrary $y \in |\mathcal{Y}|$ and an analytic path p such that $p(0) = y$, and $p(t) \in \mathfrak{X}\setminus\eta^{-1}(\Sigma)$ for $t \neq 0$. We have to prove that

$$\lim_{t\to 0} \tilde{\alpha}(p(t)) = \lim_{t\to 0} \alpha(\eta\circ p(t)) = 1.$$

Let $q = \eta\circ p$. Then $q(0) = 0$ and for $t \neq 0$

$$q(t) \in X \setminus \Sigma \ , \quad \frac{dq}{dt}(t) \in T_{q(t)}X.$$

Therefore

$$\frac{|(q(t), \frac{dq}{dt}(t))|}{\|q(t)\| \, \|\frac{dq}{dt}(t)\|} \leqq \alpha(q(t)) \leqq 1.$$

If we write

$$q(t) = at^r + (\text{higher order terms}) \quad 0 \neq a \in \mathbb{C}^N,$$

then

$$\frac{dq}{dt}(t) = rat^{r-1} + (\text{higher order terms}),$$

and the left hand side of the above inequality is equal to

$$\frac{r\|a\|^2 t^{2r-1}}{\sqrt{\|a\|^2 t^{2r} + \cdots} \ \sqrt{r^2 \|a\|^r t^{2r-2} + \cdots}} = 1 + \varphi(t)$$

with $\varphi(0) = 0$.

Consequently

$$1 + \varphi(t) \leq \alpha(q(t)) \leq 1$$

and $1 = \alpha(q(0))$. Q.E.D.

(Once we notice that the problem can be expressed by limits along analytic paths then everything become trivial.)

Theorem 2.5. If ℓ in Remark 2.3 (2) is sufficiently general, then T is the tangent of the reduced tangent cone along ℓ.

Remark 2.6. The tangent cone is deformation of the corresponding analytic space.

Case 1. Let $X \subset \mathbb{C}^{n+1}$ be a hypersurface defined by $f=0$. Let

$$f = f_m + f_{m+1} + \cdots$$

be the Taylor expansion of f, where f_k is a homogeneous polynomial of degree k and $f_m \neq 0$. We set

$$\begin{aligned} F(x,t) &= f_m(x) + tf_{m+1}(x) + t^2 f_{m+2}(x) + \cdots \\ &= \frac{1}{t^m} f(tx). \end{aligned}$$

Let $Z \subset \mathbb{C}^{n+1} \times \mathbb{C}$ be the analytic space defined by $F = 0$, $\varphi : Z \to \mathbb{C}$ be the map induced by the projection $\mathbb{C}^{n+1} \times \mathbb{C} \to \mathbb{C}$. Then, $\varphi^{-1}(0) \cong C_{X,0}$ and $\varphi^{-1}(t) \cong X$ if $t \neq 0$. Since the element t is not a zero-divisor in $\mathcal{O}_Z$, the map φ is flat.

Case 2. Let $X \subset \mathbb{C}^N$ be a arbitrary analytic space. We can choose $f_1,\cdots,f_k$ such that $in_0 f_1,\cdots, in_0 f_k$ define $C_{X,0}$. We set

$$F_i(x,t) = \frac{1}{t^{m_i}} f_i(tx) \qquad i=1,2,\cdots,k$$

where $m_i = \text{ord } f_i$.

Let $Z \subset \mathbb{C}^N \times \mathbb{C}$ be the analytic space defined by

$$F_1 = F_2 = \cdots = F_k = 0,$$

$\varphi : Z \to \mathbb{C}$ be the map induced by the projection to the t-axis.

One can show that the element t is a non-zero divisor in $\mathcal{O}_{Z,(0,0)}$. Thus $\varphi : Z \to \mathbb{C}$ is flat.

<u>Definition 2.7.</u> The family of <u>smooth</u> analytic subset S_i $i \in I$ of a analytic space X is called a <u>stratification</u> of X if the following conditions are satisfied.

(1) $X = \coprod_{i \in I} S_i$ (disjoint union) where I is a finite set.

(2) $\bar{S}_i - S_i$, $\bar{S}_i$ are analytic subsets.

(3) If $S_i \cap \bar{S}_j \neq \emptyset$, then $S_i \subset \bar{S}_j$.

Each S_i is called a <u>stratum</u>. The condition (3) is equivalent to the next (3')

(3') $\bar{S}_i$ is union of strata.

Let $\varphi : (Z,0) \to (\mathbb{C},0)$ be a analytic map, and $Z = \coprod_{i \in I} S_i$ be a stratification. We assume that the rank of φ is constant on each stratum. We say that the stratification $Z = \coprod_{i \in I} S_i$ satisfies the <u>Thom condition</u> if the following condition is satisfied: Let S_i, S_j be a pair of strata such that $S_j \subset \bar{S}_i$. Let $z_k \in S_i$ be a sequence of points of S_i such that

(1) it converges to a point $z \in S_j$

(2) the sequence of tangent spaces $T_{z_k}(\varphi^{-1}\varphi(z_n) \cap Z_i)$ have a limit T.

Then, we have

$$T \supset T_z(\varphi^{-1}(\varphi(z)) \cap S_j).$$

Remark 2.8. In [6], H. Hironaka shows that under the condition that for every $t \in \mathbb{C}$ $\varphi^{-1}(t)$ is contained the closure of $Z - \varphi^{-1}(t)$ in Z, there exists a stratification such that

(1) the rank of φ is constant on each stratum

(2) it satisfies the Thom condition

(3) Each stratum is contained $\varphi^{-1}(0)$ or $Z - \varphi^{-1}(0)$.

The proof of Theorem 2.5. Let $\varphi : Z \to \mathbb{C}$ be the map described in Remark 2.6. Case 2. It is easy to see that because of flatness for every $t \in \mathbb{C}$, $\overline{Z-\varphi^{-1}(t)} \supset |\varphi^{-1}(t)|$. By Remark 2.8, we know that there exists a stratification which satisfies the above conditions (1), (2), (3).

Let $U \subset |\varphi^{-1}(0)|$ be union of the stratum which is contained $|\varphi^{-1}(0)| = |C_{X,0}|$ and which has the same dimension as $|C_{X,0}|$. U is dense in $|C_{X,0}|$.

Let $\xi_k \in X \setminus \Sigma$ be an arbitrary sequence such that

(i) $\lim \xi_k = 0$

(ii) there exists the limit $T = \lim T_{\xi_n} X$

(iii) there exists the limit $\ell = \lim \overline{\xi_n 0} \subset C_{X,0}$

(iv) moreover $\ell \cap U \neq \emptyset$.

The condition (iii) implies that for the sequence $\xi_k = (\xi_{1k}, \xi_{2k}, \cdots, \xi_{N,k}) \in \mathbb{C}^N$, there exists an index α such that $\xi_{\alpha,k} \neq 0$ for sufficiently large k and that the sequence

$$\tilde{\xi}_k = (\xi_{1k}/\xi_{\alpha k}\ ,\ \xi_{2k}/\xi_{\alpha k}, \cdots, \xi_{Nk}/\xi_{\alpha k})$$

has a limit $\tilde{\xi} \in \mathbb{C}^N$. By multiplying a non zero element to coordinates if necessary, we may assume $\tilde{\xi} \in \ell \cap U$ by (iv).

Let $z_k = (\tilde{\xi}_k, \xi_{\alpha k}) \in \mathbb{C}^N \times \mathbb{C}$. By definition $z_k \in Z$ and $\lim z_k = (\tilde{\xi}, 0) \in U$. And $T_{z_k}(\varphi^{-1}\varphi(z_k)) = T_{\xi_k} X \times (0)$. Thus there exists a limit $T = \lim T_{z_k}(\varphi^{-1}(\varphi(z_k)))$. Then, by the Thom condition

$$T \supset T_{(\tilde{\xi},0)}|\varphi^{-1}(0)| = T_{\tilde{\xi}}|C_{X,0}|.$$

This inclusion implies $T = T_{\tilde{\xi}}|C_{X,0}|$ because these spaces have the same dimension. Q.E.D.

§3. Surfaces

In this section we apply our general results to surfaces in $\mathbb{C}^3$. (Cf. Lê-Teissier [9]).

Let $(X,0) \subset (\mathbb{C}^3,0)$ be a surface defined by $f(x,y,z) = 0$, where f is a holomorphic function on a open neighbourhood U of the origin 0.

Theorem 3.1.

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\tilde{e}"] \arrow[d, "\nu'"'] \arrow[dr, "\eta"] & \tilde{X} \arrow[d, "\nu"] \\ X' \arrow[r, "e"'] & X \end{tikzcd}.$$

We consider the diagram explained in §2. We denote $Y' = e^{-1}(0)$, $\tilde{Y} = \nu^{-1}(0)$, and $\mathcal{Y} = \nu'^{-1}(Y') = \tilde{e}^{-1}(\tilde{Y}) = \eta^{-1}(0)$. Note that $\dim \mathcal{Y} = \dim Y' = 1$.

(1) $\nu' : |\mathcal{Y}| \to |Y'|$ is generically one to one. Let $\{\ell_1, \ell_2, \cdots, \ell_k\}$ be the set of the point ℓ such that $\dim \nu'^{-1}(\ell) = 1$. Each ℓ_i can be regarded as a generatrix of the tangent cone $C_{X,0}$.

(2) For each $i = 1,2,\ldots,k$, $\nu'^{-1}(\ell_i)$ can be regarded as the pencil of planes containing ℓ_i.

(3) The components $\mathcal{Y}_1, \ldots, \mathcal{Y}_\ell$ of $|\mathcal{Y}|$, which are finite over $|Y'|$ are in one-to-one correspondence with the components $Y'_1, \cdots, Y'_\ell$ of $|Y'|$.

(4) $|\tilde{e}(\mathcal{Y}_j)|$ is the dual variety of Y'_j.

<u>Definition 3.2.</u> Generatrices $\ell_1, \ell_2, \cdots, \ell_k$ are called <u>exceptional tangents</u> of X at 0.

<u>Proof of Theorem 3.1.</u> (1),(3),(4) are easy consequences of Theorem 2.5.

(2) follows from the Whitney lemma.

Q.E.D.

Next we introduce some notions.

<u>Definition 3.3.</u> Let $P_L : X^0 = X - \Sigma \to \mathbb{C}^2$ be the projection along the line L. The closure $\overline{C(P_L)}$ of the critical locus $C(P_L)$ of P_L is called <u>a polar curve</u> associated with

the line L, if it is an analytic set and has dimension 1,

Remark 3.4. Let $C_L \subset G(2,3) = \mathbb{P}^{2\vee}$ be the set of planes containing the line L. We have

$$\overline{C(P_L)} = \overline{\nu(\gamma^{-1}(C_L)-\nu^{-1}(0))}.$$

Let (x,y,z) be the coordinate of $\mathbb{C}^3$ such that L is defined by $x = y = 0$. $\overline{C(P_L)}$ is the closure $\overline{\{f = 0, \frac{\partial f}{\partial z} = 0\} \setminus \text{singular locus } \Sigma}$.

The following Proposition 3.5 is a consequence of the preceding results. (See Henry-Lê [4], Lê [8], Lê-Teissier [9].)

Proposition 3.5. Let $\rho : Z \to Y$ be a flat map of reduced analytic spaces such that for every point $y \in Y$, the fibre $Z_y = \rho^{-1}(y)$ has dimension 1 and Z_y is reduced. We assume for every $y \in Y$, Z_y has imbedding dimension 2 at every point of $z \in Z_y$. For given points $y \in Y$ and $z \in \rho^{-1}(y)$, the following conditions are equivalent.

(1) There exists an open neighbourhood U of z in Z, such that denoting the open set $\rho(U)$ by B and denoting, for every $y' \in B$ the singular points of the curve $Z_{y'} \cap U$ by z_i' $(1 \leq i \leq k)$, we have

$$\mu(Z_y,z) = \sum_{i=1}^{k} \mu(Z_{y'};z_i),$$

where $\mu(Z_y,z)$ denotes the Milnor number of Z_y at z.

(2) There exists an open neighbourhood U of z in Z and a section $\sigma : B \to U$ of ρ, such that

α) ρ induces a submersion of non-singular variety

$$U - \sigma(B) \to B$$

β) We have

$$\mu(Z_{y'};\sigma(y')) = \mu(Z_y,z)$$

for every $y' \in B$.

(2') The same conditions as in (2), but in addition to those,

γ) We have

$$m(Z_{y'},\sigma(y')) = m(Z_y,z)$$

where $m(Z_y,z)$ denotes the multiplicity of the maximal ideal of the local ring $\mathcal{O}_{Z_y,z}$.

(3) There exists an neighbourhood U of z in Z and a section $\sigma : B \to U$ of ρ, such that for every $y' \in B$, the topological type of the germ of the plane curve $(Z_{y'};\sigma(y'))$ is constant.

(4) There exist a open neighbourhood U of z in Z, a section $\sigma : B \to U$ with $B = \rho(U)$ and a projection $\eta : B \times \mathbb{C}^2 \to B \times \mathbb{C}$ such that

α) One has a local embedding

$$i : (U,z) \to (Y \times \mathbb{C}^2, y \times 0)$$

which makes the next diagram commutative.

$$\begin{tikzcd} U \arrow[rr, "i"] \arrow[dr, "p"'] & & B \times \mathbb{C}^2 \arrow[dl, "pr_1"] \\ & B & \end{tikzcd}$$

and for which one has $i(\sigma(B)) = B \times \{0\}$.

β) The ramification locus of the restriction of η to U coincides with $B \times \{0\}$.

If one of the above conditions is satisfied, we say that Z is <u>equisingular</u> along $\sigma(B)$ at z, where σ is the section appearing in (2), (2') and (3).

Now we can state a theorem.

<u>Theorem 3.6.</u> (Characterization of exceptional tangents.) Let $(X,0)$ be a germ of analytic surface in $(\mathbb{C}^3,0)$ defined by $f = 0$. Let ℓ be a generatrix of the tangent cone $C_{X,0}$ of X at 0, <u>which is not tangent to the singular locus Σ</u>.

The following assertions are equivalent.

(1) The generatrix ℓ is an exceptional tangent of X at 0.

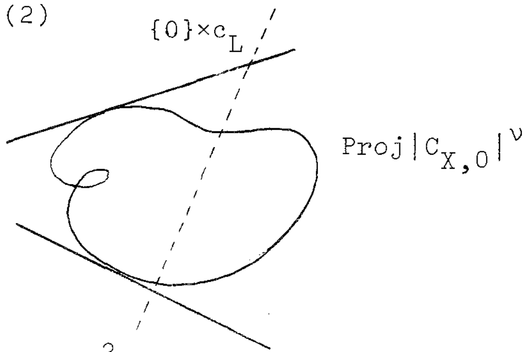
(2) For every local projection $P_L : (X,0) \to (\mathbb{C}^2,0)$ which has the same degree as the multiplicity in $m(X,0)$ of X at 0, the generatrix ℓ is tangent to the polar curve $\overline{C(P_L)}$.

(3) The surface X' is not equisingular along $|e^{-1}(0)| = \mathrm{Proj}|C_{X,0}|$ at the point of $e^{-1}(0)$ which represents ℓ.

(4) Let $\varphi : Z \to \mathbb{C}$ be the deformation defined in Remark 2.6.

Let Z_0 be the smooth part of $Z \setminus \varphi^{-1}(0)$, Z_1 be the smooth part of $|\varphi^{-1}(0)| = |C_{X,0}| \times \{0\}$. Then, the pair (Z_1, Z_0) of strata do not satisfy the Thom condition at every point of $\ell \times \{0\} \setminus 0$.

<u>Proof.</u> (1) $\Rightarrow$ (2)



Let $P_L : (X,0) \to (\mathbb{C}^2,0)$ be a projection as in (2), where L denotes the line of the center of the projection. Let c_L be the pencil of planes containing L. Since $\gamma\tilde{e}(\nu'^{-1}(\ell))$ is a line in $G(2,3) \cong \mathbb{P}^{2\vee}$, $\gamma\tilde{e}(\nu'^{-1}(\ell))$ is necessarily intersects with c_L. Thus, $\emptyset \neq \tilde{e}(\nu'^{-1}(\ell)) \cap \gamma^{-1}(c_L) = \tilde{e}(\nu'^{-1}(\ell)) \cap \overline{\gamma^{-1}(c_L) \setminus \tilde{Y}}$. Here we note that $\gamma^{-1}(c_L) = \overline{\gamma^{-1}(c_L) \setminus \tilde{Y}}$ because if a certain component of $\gamma^{-1}(c_L)$ is contained in $\tilde{Y} \subset \{0\} \times \mathbb{P}^{2\vee}$, it should be a linear component, it coincides with $\nu'^{-1}(\ell_i)$ for some exceptional tangent ℓ_i, and we have $L = \ell_i$, which contradicts to the choice of L. We have

$$\nu'^{-1}(\ell) \cap \overline{\tilde{e}^{-1}\gamma^{-1}(c_L) \setminus \mathcal{Y}} \neq \emptyset,$$

and

$$\ell \in \overline{\nu'\tilde{e}^{-1}\gamma^{-1}(c_L) \setminus Y'} = \Gamma'.$$

However since Γ' is the strict transform of the polar curve

$\Gamma = \overline{C(P_L)}$, $e : \Gamma' \to \Gamma$ in the blowing up of the maximal ideal of $\mathcal{O}_{\Gamma,0}$. Thus it follows that the generatrix ℓ is tangent to the polar curve Γ.

(2) $\Rightarrow$ (1)

Let ℓ be a generatrix of $C_{X,0}$ which is not tangent to the singular locus Σ and which is not an exceptional tangent. By this assumption, $\gamma\tilde{e}(\nu'^{-1}(\ell))$ is a finite set. Thus, for the general line L, c_L does not pass through any point of $\gamma\tilde{e}(\nu'^{-1}(\ell))$, which shows that $\overline{C(P_L)}$ is not tangent to ℓ for a general projection P_L.

In order to see the equivalence between (1) and (4), we for a moment assume that ℓ corresponds to a smooth point of $|Y'| = \mathrm{Proj}|C_{X,0}|$, in other words $\ell \times \{0\} \setminus 0 \subset Z_1$. The proof of Theorem 2.4 shows that (1) implies (4). Conversely we assume that (1) does not hold. Generatrix ℓ is not exceptional. Let $(z_n, t_n) \in Z_0$ be a sequence which converges to $\ell \in Z_1$, we set $x_n = t_n z_n \in X - \Sigma$. The sequence $\xi_n = \tilde{e}^{-1}\nu^{-1}(x_n)$ converges to a point $\xi \in \mathcal{Y}$. By the choice of the sequence $\xi \in \nu'^{-1}(\ell)$. By Theorem 3.1, $\nu'^{-1}(\ell)$ consist of one element $\{(\ell, T)\}$, where T is the tangent space of the cone $|C_{X,0}|$ along ℓ. Since $T_{(z_n,t_n)}(\varphi^{-1}(t_n) \cap Z_0) = T_{x_n} X$, we know that the Thom condition holds for (Z_1, Z_0) at ℓ.

As for (3) and the more precise proof on (4), see Lê [8].

Q.E.D.

Corollary 3.7. The tangent cone of the polar curve Γ for a generic linear space L is given by

(1) the exceptional tangents

(2) The critical locus of the induced projection

$$dP_L : |C_{X,0}| \to \mathbb{C}^2.$$

Proof. By Theorem 3.1 and the proof of Theorem 3.6, it is obvious.

Q.E.D.

Corollary 3.7. If $|C_{X,0}|$ is a plane and the projection P is transversal to that plane, each tangent of the discriminant of P is the image of an exceptional tangent.

Proof. The discriminant is the image of polar curve. Thus, it is obvious by Corollary 3.6.

Q.E.D.

Example 3.8. Let X be a surface defined by a Pham-Brieskorn polynomial $x^a + y^b + z^c = 0$. We first assume that $a < \inf(b,c)$.

The tangent cone $|C_{X,0}|$ is given by $x = 0$. The discriminant of the projection $(x,y,z) \to (y,z)$, which is transversal to this plane $x = 0$, is given by $y^b + z^c = 0$.

① $a < b < c$

the line $x = y = 0$ is the only exceptional tangent.

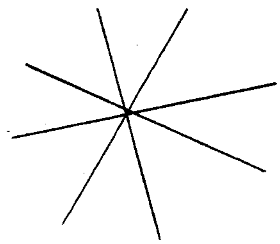
② $a < b = c$

$$y^b + z^b = (y+\varepsilon z)(y+\varepsilon^3 z)(y+\varepsilon^5 z)\cdot\cdots\cdot(y+\varepsilon^{2b-1}z)$$

$$\varepsilon = \exp\left(\frac{2\pi\sqrt{-1}}{2b}\right).$$

We have b exceptional tangents

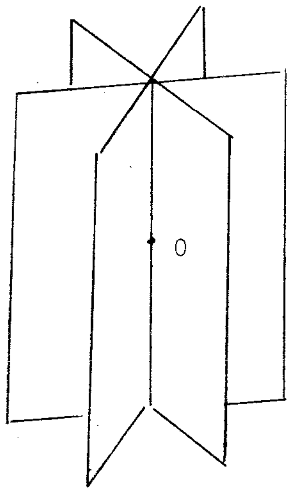
$$\begin{cases} x = 0 \\ y + \varepsilon^{\alpha} z = 0 \qquad \alpha = 1,3,5,\cdots,2b-1. \end{cases}$$



Moreover,

(3) $a = b < c$

The tangent cone is given by $x^a + y^a = 0$.



The axis of the tangent cone, $x = y = 0$ is the only one exceptional tangent.

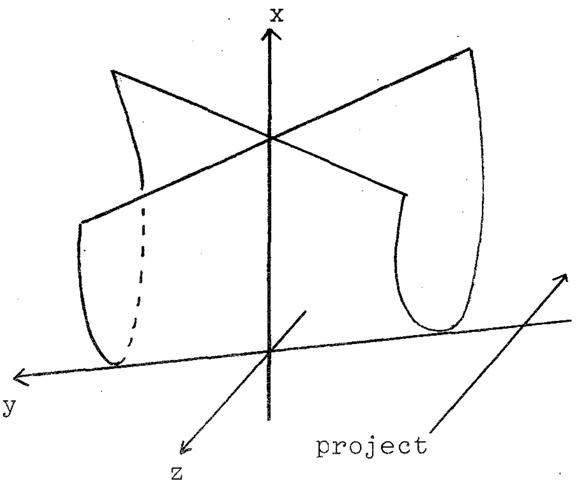
(4) $a = b = c$

$X : x^a + y^a + z^a = 0$ has no exceptional tangent.

<u>Example 3.9</u>. The Whitney umbrella is given by

$$xy^2 - z^2 = 0.$$

The singular locus is the x-axis.



The projection $p : (x,y,z) \to (x,y)$ is transversal to the tangent cone $|C_{X,0}|$ which is given by $z = 0$. The critical locus $C(P)$ of P is given by

$$xy^2 - z^2 = 0, \quad \frac{\partial(xy^2-z^2)}{\partial z} = -2z = 0.$$

Thus the polar curve $\overline{C(P)\setminus\Sigma} = \Gamma$ is defined by $x = z = 0$, which is the y-axis.

The critical set of a general projection is given by $f = xy^2 - z^2 = 0$ and

$$a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = ay^2 + 2bxy - 2z = 0.$$

Since

$$xy^2 - (\frac{a}{2}y^2 + bxy)^2 = y^2\{x - (\frac{a}{2}y + bx)^2\},$$

the equation of polar curve Γ is

$$x - (\tfrac{a}{2}y + bx)^2 = 0$$

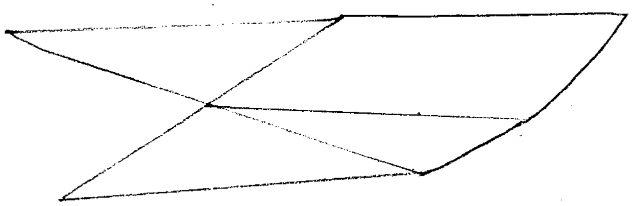
$$ay^2 + 2bxy - 2z = 0.$$

We note that the tangent line of Γ, $x = z = 0$, does not depend on the parameter. Thus the line $x = z = 0$ is an exceptional tangent.

Example 3.10. The swallow tail X, discriminant of the semi universal deformation of $u^4 = 0$ is given by

$$256x^3 - 27y^4 - 128x^2z^2 + 144xy^2z + 16xz^2 - 4y^2z^3 = 0.$$

The tangent cone $|C_{X,0}|$ is a plane $x = 0$. However, X does not have exceptional tangents, since we know that the Nash modification $\nu : \tilde{X} \to X$ is a finite map. (See B. Teissier [14].)



In the remaining part of this section we shall state some theorems without proof. Readers can find the proof in Lê [8], Lê-Teissier [9].

Definition 3.11. Let $Z = \coprod_{i \in I} S_i$ be a stratification of a analytic space Z. We say that the Whitney condition is satisfied for this stratification if the following condition is satisfied:

Let S_i, S_j be strata such that $S_i \subset \overline{S_j}$. Let $x_n \in S_i$ and $y_n \in S_j$ be sequences of points such that

(1) $\lim y_n = \lim x_n = x \in S_i$

(2) There exists a limit $T = \lim T_{y_n} S_j$

(3) Lines $\overline{x_n y_n}$ have a limit $\ell = \lim \overline{x_n y_n}$.

Then, we have $\ell \subset T$.

Remark 3.12. For any germ $(X,0)$ of analytic space, there exists a representant $X \subset \mathbb{C}^N$ in some $\mathbb{C}^N$, which can be stratified with the Whitney condition. (See H. Whitney [18] or H. Hironaka [6]).

Theorem 3.13. (Cf. Lê [8]). Let $(X,0) \subset (\mathbb{C}^3,0)$ be a surface. Suppose the tangent cone $C_{X,0}$ is reduced. Then the following two conditions are equivalent.

(1) The surface X has no exceptional tangents.

(2) The deformation $\varphi : Z \to \mathbb{C}$ of the tangent cone defined in Remark 2.6 is equisingular at 0 along $\{0\} \times \mathbb{C}$ in one of the following senses.

1 The non-singular part of the space $\hat{Z}$ obtained by blowing-up of $\{0\} \times \mathbb{C}$ in Z satisfy the Whitney condition along its singular part.

② The stratification $(Z \setminus \mathrm{Sing}\, Z,\ \mathrm{Sing}\, Z - \{0\} \times \mathbb{C},\ 0 \times \mathbb{C} \cap Z)$ satisfies the Whitney condition.

③ For a general projection $\pi : \mathcal{X} \to \mathbb{C}^2 \times \mathbb{C}$ which is compatible with φ and the projection $\mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}$, the discriminant Δ of π is the family of plane curves which is equisingular along $\{0\} \times \mathbb{C}$.

Theorem 3.13. Let $(X,0) \subset (\mathbb{C}^3,0)$ be a germ of surface with an isolated singularity. The following conditions are equivalent.

(1) The blowing up $e : X' \to X$ of the origin is equisingular along the curve $|e^{-1}(0)|$.

(2) The tangent cone $C_{X,0}$ of X at 0 is reduced and has no singular point outside of 0.

(3) The Milnor number $\mu^{(3)}$ of $(X,0)$ and the Milnor number $\mu^{(1)} = m(X,0) - 1$ of its section with a general line satisfy an equality

$$\mu^{(3)} = (\mu^{(1)})^3.$$

(4) The deformation $\varphi : Z \to \mathbb{C}$ of the tangent cone $C_{X,0}$ defined in Remark 2.6 is equisingular in the sense that the family $\varphi : Z \to \mathbb{C}$ has a simultaneous resolution.

(5) The blowing up $e : X' \to X$ of the maximal ideal at the origin is a resolution of singularities and the curve $(e^{-1}(0))$ is non-singular.

(6) The surface X has no exceptional tangents at the origin.

§4. Higher Dimensional Cases

Let $(X,0) \subset (\mathbb{C}^N,0)$ be a reduced germ of analytic spaces of pure dimension d.

Let $P_k: \mathbb{C}^N \to \mathbb{C}^{1+k}$ be a projection i.e. a surjective linear map. We denote $\tilde{P}_k = P_k|_{X^0}: X^0 \to \mathbb{C}^{1+k}$.

Definition 4.1. The closure of the critical locus $\Gamma_k = \overline{C(\tilde{P}_k)}$ of $\tilde{P}_k$ is called the k polar varieties associated with the projection $P_k : \mathbb{C}^N \to \mathbb{C}^{1+k}$.

Remark 4.2.

1. All projections $\mathbb{C}^N \to \mathbb{C}^{1+k}$ constitute a Grassmann variety $G(N-k-1, N) = G$. We can show that for a non-empty Zariski open set $U \subset G$, and for every projection $P \in U$, the polar variety associated to P is empty or it is reduced of dimension k. See Lê-Teissier [10].

   In the following we discuss only projections belonging to this open set U. We call them generic ones.
2. If $k = d$ $\Gamma_d = X$.
3. If X is smooth, then $\Gamma_k = \emptyset$ unless $k = \dim X$.
4. We can associate to $0 \in X$, d-uple of integers

$$e(X,0) = (m_0(\Gamma_1), m_0(\Gamma_2), \cdots, m_0(\Gamma_d))$$

where $m_0(\Gamma_k)$ denotes the multiplicity of the generic k-polar variety at 0. Then, $e(X,0) = (0,0,\cdots,0,1) \Leftrightarrow (X,0)$ is non-singular. We can see that the correspondance

$$X \ni x \longmapsto e(X,x)$$

is a constructible semi-continuous map.

5. Let $\nu : \tilde{X} \to X$ be the Nash modification and $\gamma : \tilde{X} \to G(d,N)$ be the extended Gauss map.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\gamma} & G(d,N) \\ {\scriptstyle\nu}\big\downarrow & & \\ X & & \end{array}$$

We set

$$C_{d-k} = \{T \in G(d,N) \mid \begin{array}{l} T \text{ and } H_{k+1} \text{ does not span the} \\ \text{ambient space } \mathbb{C}^N \end{array}\}$$

where H_{k+1} is a fixed linear subspace of $\mathbb{C}^N$ of codimension $k+1$.

By definition, we have

$$\Gamma_k = \nu(\gamma^{-1}(C_{d-k})),$$

where Γ_k is the polar variety associated to the projection such that its kernel coincides with H_{k+1}.

Let $(X,0) \subset (\mathbb{C}^{d+1},0)$ be a reduced germ of analytic hypersurfaces of dimension d. Surprisingly the situation is like the one of a surface.

The notations are the same as in §2.

$$\begin{tikzcd}
|\nu^{-1}(0)| = |\mathcal{Y}|^{d-1} \subset \mathfrak{X} \arrow[r, "\tilde{e}"] \arrow[d, "\nu"] & \tilde{X} \arrow[d, "\nu'"] \\
\mathrm{Proj}|C_{X,0}| = |e^{-1}(0)| = |Y'|^{d-1} \subset X' \arrow[r, "e"] & X .
\end{tikzcd}$$

Every components of $|\mathcal{Y}|$ has dimension d-1 because $\mathcal{Y}$ is a Cartier divisor.

Theorem 4.3. Let $|\mathcal{Y}| = \bigcup_{\alpha\in I} \mathcal{Y}_\alpha$ be the decomposition of $|\mathcal{Y}|$ into irreducible components. Let $\{\mathcal{Y}_\alpha | \alpha \in I_0\}$ be the components which are generically finite over Y'.

(1) $Y' = \bigcup_{\alpha\in I_0} Y'_\alpha$

with $Y'_\alpha = \nu'(\mathcal{Y}_\alpha)$ is the decomposition of $|Y'|$ into irreducible components. In particular, the set $\{\mathcal{Y}_\alpha | \alpha \in I_0\}$ and the irreducible components of $|Y'|$ are in one to one correspondence.

(2) The set $|\tilde{Y}|$ of limit tangents of X at 0 is the union of the dual variety V^*_α of $V_\alpha = \nu'(\mathcal{Y}_\alpha)$, where α runs over the whole set I.

Remark 4.4. For a projective variety $V \subset \mathbb{P}^{N-1}$, the dual variety V^* is defined as the closure of the set of hyperplanes which are tangent to V. We say that a hyperplane H is tangent to V, if H contain a tangent space T_xV at a smooth point $x \in V$. V^* is a subvariety of the dual projective space $\mathbb{P}^{N-1\vee}$. It is known that the dual V^{**} of the dual variety coincides with the original V.

In the next theorem $\hat{V}$ denotes the affine cone over the projective variety $V \subset \mathbb{P}^{N-1}$.

Theorem 4.5. The tangent cone of the generic k-polar variety Γ_k is the union of k-polar variety of $\hat{V}_\beta$, where $V_\beta = \nu'(\mathcal{Y}_\beta)$.

Remark 4.6. If $\dim \hat{V}_\beta < k$, the k-polar variety of $\hat{V}_\beta$ is empty. Thus the union in Theorem 4.5 can be taken only over V_β such that $\dim V_\beta \geq k$.

The proof of Theorem 4.3, Theorem 4.5 is done by induction on the dimension. We don't give it here. However, the ideas are the same as in the case of surfaces plus the following.

(1) Let $(X,0) \subset (\mathbb{C}^{N+1},0)$ be a reduced germ of analytic space of pure dimension d, which is not necessarily a hypersurface.

Let $(X,0) \to (\mathbb{C}^{d+1},0)$ be a generic projection and $(X_1,0)$ be its image. Then, the image of polar varieties are the polar varieties of the image. And the multiplicity doesn't change under this projection.

(2) There is an unhappy fact. There is no canonical map from the Nash modification $\tilde{X}$ of X to that $\tilde{X}_1$ of X_1.

Theorem 4.7. (Cf. Lê-Teissier [10] appendix). Let $(X,0) \subset (\mathbb{C}^N,0)$ be a reduced germ of an analytic space of pure dimension d.

For any generic projection $P : \mathbb{C}^N \to \mathbb{C}^{d+1}$, we have:

(1) The induced map from X to $X_1 = P(X)$ by P is finite and bimeromorphic.

(2) In the Grassmann variety $G = G(d,N)$ let C be the Schubert variety of d-planes which are not transversal to Ker P. Then the intersection of C and each irreducible component Y_k of $Y = \nu^{-1}(C)$ has codimension 2 in Y_k or else it is empty.

(3) The induced map $\tilde{P} : X \to X_1$ by P can be extended to a map : $\tilde{X} - \gamma^{-1}(C) \to \tilde{X}_1$, which is finite.

Proof. (1) is classical. (2) follows from the fact that C has codimension 2 in G and Kleiman's general position theorem. As for (3), the existence of the extension follows by linear algebra. The only thing to prove is the finiteness of the extended map. Readers can find in Lê-Teissier [10] a precise proof.

Proposition 4.8. Let $(X,0)$ be a reduced germ of an analytic space of pure dimension d, $\nu : \tilde{X} \to X$ be the Nash modification of sufficiently small representant of $(X,0)$. One has $\dim \nu^{-1}(0) < d-k$ if and only if $\Gamma_k = \emptyset$ where Γ_k is the k-polar variety of X associated to a generic projection $\mathbb{C}^N \to \mathbb{C}^{1+k}$.

First we assume that X is a hypersurface. Then we can prove this proposition by the Lê-Teissier formula and topological arguments. Next by Theorem 4.7, we reduce the general case to hypersurface one.

(3) The next theorem is useful when we use induction on the dimension.

Theorem 4.9. In the same situation as in Theorem 4.7, let $(H,0) \subset (\mathbb{C}^N,0)$ be a hyperplane and let C_H be the Schubert subvariety in $G = G(d,N)$ consisted of d-planes contained in H. We assume

$$\gamma^{-1}(C_H) \cap \nu^{-1}(0) = \emptyset.$$

Then, we have the map $(X \cap H)^0 \to G(d-1,N)$ which associates $x \in (X \cap H)^0$ to $T_{X\cap H,x} = T_{X,x} \cap H$ can be extended to a finite and bimeromorphic map $(X \cap H)\hat{} \to \widetilde{X \cap H}$ from the strict transform of $X \cap H$ by ν to the Nash modification of $X \cap H$.

The finiteness of the extended map is not obvious. The other part is proved by standard argument. As for the precise proof see Lê-Teissier [10].

Remark 4.10. It is easy to check that for a generic hyperplane H, the assumption of Theorem 4.9 $\gamma^{-1}(C_H) \cap \nu^{-1}(0) = \emptyset$ is satisfied.

Chapter II. Singular Chern Class

In this chapter, we shall study the relation between geometry of tangents and MacPherson's theory of singular Chern classes.

§1. Review of Obstruction Theory

First we recall basic results of obstruction theory, following the Steenrod's book "The topology of fibre bundles" [13].

Let K be a cell complex of dimension n. Let $\mathcal{B} \xrightarrow{\pi} K$ be a fibre bundle over K with fibre F. Let $L \subset K$ be a subcomplex and suppose we have a section of $\mathcal{B} \to K$ defined on L.

<u>Problem</u>. Can we extend it to K?

Let

$$K_0 \subset K_1 \subset \dots \subset K_n = K$$

be the skeletons of K. K_j is the union of cells of K of dimension less than or equal to j. For a given section $s_L: L \to \mathcal{B}$, we can easily extend it to $s_0: L \cup K_0 \to \mathcal{B}$: For $x \in L \cup K_0 \setminus L$ we can choose any arbitrary point on the fibre as the value $s_0(x)$, since $L \cup K_0 \setminus L$ consists of discrete points.

$$\begin{array}{ccccc} \mathcal{B} & & \mathcal{B} & & \\ \nearrow s_L & & \nearrow s_0 & & \\ L & \subset & L \cup K_0 & \subset & L \cup K_1 \end{array}$$

If we have an extension $s_1: L \cup K_1 \to \mathcal{B}$, for a 1-cell $\sigma \subset K_1 \setminus L$, the restriction $s_1|_{\bar{\sigma}}$ extends $s_1|_{\partial\sigma} = s_0|_{\partial\sigma}$. Thus a sufficient condition to get s_1 is that F is arcwise connected more precisely:

Let $\phi_\sigma: \sigma \times F \to \mathcal{B}|_\sigma = \pi^{-1}(\sigma)$ be an isomorphism which makes the following diagram commutative: Such an isomorphism exists, since σ is contractible.

$$\begin{tikzcd} \mathcal{B}|_\sigma \arrow[dr, "\pi|_\sigma"'] & & \sigma \times F \arrow[ll, "\phi_\sigma"'] \arrow[dl, "p_1"] \\ & \sigma & \end{tikzcd}$$

Here $p_1: \sigma \times F \to \sigma$ is the projection onto the first factor. Let $\tilde{s}: \partial\sigma \to F$ be the composition

$$\partial\sigma \xrightarrow{s_0|\partial\sigma} \mathcal{B}|_\alpha \xrightarrow{\phi_\alpha^{-1}} \sigma \times F \xrightarrow{p_2} F$$

where p_2 is the projection to the second factor. If F is arcwise connected, we can extend $\tilde{s}$ to $\tilde{s}_\sigma$. Then

$$s_\sigma(x) = \phi_\sigma(x, \tilde{s}_\sigma(x)) \qquad x \in \sigma$$

gives an extension of $s_0|_{\partial\sigma}$ and we have $s_1: L \cup K_1 \to \mathcal{B}$ by setting

$$s_1|_\sigma = s_\sigma \quad \text{for a 1-cell } \sigma \subset K_1 \setminus L.$$

Obviously we can procceed by induction on the dimension of cells.

Suppose F is (q-1)-connected. That is, F is arcwise connected and $\pi_i(F,x) = 0$ for every integer i with $1 \le i \le q-1$. Then, any section s_L on L an be extended to the q-th skeleton $L \cup K_q$.

Suppose for some q, $\pi_q(F,x) \neq 0$ and F is (q-1)-connected. Let σ be a (q+1)-cell of K such that $\sigma \not\subset L$. The extension $s_q: L \cup K_q \to \mathcal{B}$ defines a class

$$[s_q|_{\partial\sigma}] \in \pi_q(F).$$

Then, the collection $\{[s_q|_{\partial\sigma}]\}_\sigma$ and some nice conditions on F will define a cohomology class for some good cohomology.

However we have to check some further points.

(1) We have to eliminate the hase point problem. The choice of the hase point of $x \in F$ does not matter if F is q-simple that is, $\pi_1(F,x)$ operates trivially on $\pi_i(F,x)$ for $0 \leq i \leq q$.

Note that F is q-simple for any q if $\pi_1 = 0$.

(2) The cohomology should be the cohomology of (K,L) with value in a sheaf $\mathcal{B}(\pi_q)$ of abelian groups defined by associating $\pi_q(\pi^{-1}(U))$ to every open set $U \subset K$.

Then, we have the obstruction class in $H^{q+1}(K,L;\mathcal{B}(\pi_q))$.

This depends only on the homotopy class of $s_L : L \to \mathcal{B}$.

In the case that $\mathcal{B}$ is an orientable vector bundle of rank r minus the zero section, our cohomology is the usual cohomology

$$H^r(K,L;\mathbb{Z}).$$

§2. Euler Obstruction

Let $(X,0) \subset (\mathbb{C}^N,0)$ be a germ of reduced analytic space of pure dimension d.

In this section we give a interpretation of number $Eu_0(X)$ which plays an important role in MacPherson's theory.

We use the notation explained in Chapter I section 2.

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\tilde{e}"] \arrow[d, "\nu"'] \arrow[dr, "\eta"] & \tilde{X} \arrow[d, "\nu"] \\ X' \arrow[r, "e"] & X \end{tikzcd}$$

$Y' = e^{-1}(0)$

$\tilde{Y} = \nu^{-1}(0)$

$\mathcal{Y} = \eta^{-1}(0)$

$\tilde{T} \to \tilde{X}$ is the Nash bundle.

Let $S_\varepsilon = \{(z_1,z_2,\dots,z_N) \in \mathbb{C}^N \mid \sum_{i=1}^{N} |z_i|^2 = \varepsilon^2\}$ and $B_\varepsilon = \{(z_1,\dots,z_n) \in \mathbb{C}^N \mid \sum |z_i|^2 \leq \varepsilon^2\}$ denote the shere and the ball with center 0 and with the radius ε. We denote a fixed hemitian form on $\mathbb{C}^N$ by $<\ ,\ >$.

Lemma 2.1. For a sufficiently small real number $\varepsilon > 0$, one has a section

$$\sigma: \nu^{-1}(S_\varepsilon \cap X) \to \tilde{T}$$

such that for $x \in \nu^{-1}(S_\varepsilon \cap X)$ $\mathrm{Re}<x,\sigma_x> > 0$ (The acute angle condition)

Remark 2.2 Every fibre of $\tilde{T} \to \tilde{X}$ can be regarded as a linear space in $\mathbb{C}^N$. Thus $<x,\sigma_x>$ has meaning. In particular the acute angle condition implies $\sigma_x \neq 0$ for $x \in \nu^{-1}(S_\varepsilon \cap X)$.

Proof of Lemma 2.1 A point $p \in \mathfrak{X}$ is a triple $(x_p,\ell_p,T_p) \in X \times \mathbb{D}^{N-1} \times G$. Consider the map

$$\varphi_p : \ell_p \to T_p$$

induced by the orthogonal projection for every point $p \in \mathfrak{X}$. By the Whitney lemma φ_p is the identity map for $p \in \mathcal{Y}$ since $\ell_p \subset T_p$. Thus for a sufficiently small $\delta > 0$, one has a neighbourhood $\mathcal{N}$ with $\mathcal{Y} \subset \mathcal{N} \subset \mathfrak{X}$ such that for every point $p \in \mathcal{N}$ and for every $v \in \ell_p$, $\|v - \varphi_p(v)\| \leq \delta \|v\|$ holds.

A section $\tau : \mathcal{N} \setminus \mathcal{Y} \to T$ is defined by $\tau(p) = \tau(x_p,\ell_p,T_p) = \varphi_p(x_p)$.

We have

$\| x_p \|^2 - \mathrm{Re}\langle x_p, \tau(p)\rangle$

$= \mathrm{Re}\langle x_p, x_p\rangle - \mathrm{Re}\langle x_p, \varphi_p(x_p)\rangle = \mathrm{Re}\langle x_p, x_p - \varphi_p(x_p)\rangle \leq \| x_p\| \cdot \delta \| x_p\|.$

Thus

$$0 < 1-\delta \leq \frac{\mathrm{Re}\langle x_p, \tau(p)\rangle}{\| x_p\|^2}$$

for $p \in \mathcal{N} \setminus \mathcal{Y}$.

Now $\tilde{e} : \mathcal{N} \setminus \mathcal{Y} \to \tilde{X} \setminus \tilde{Y}$ is an isomorphism by definition and for a sufficiently samll $\varepsilon > 0$, $\nu^{-1}(S_\varepsilon \cap X) \subset \tilde{e}(\mathcal{N} \setminus \mathcal{Y})$. We set $\sigma = \tau \circ \tilde{e}^{-1} : \nu^{-1}(S_\varepsilon \cap X) \to \tilde{T}$. Then this σ satisfies the acute angle condition. Q.E.D.

We now have an obstruction class $c_\sigma \in H^{2d}(\nu^{-1}(B_\varepsilon \cap X), \nu^{-1}(S_\varepsilon \cap X); \mathbf{Z})$ to extend σ as a non-zero section over $\nu^{-1}(B_\varepsilon \cap X)$.

<u>Lemma 2.3</u> The class c_σ does not depend on the choice of the section satisfying the actue angle condition.

<u>Proof</u>. Let σ, σ' be sections on $\nu^{-1}(S_\varepsilon \cap X)$ such that

$$\mathrm{Re}\langle x, \sigma_x\rangle > 0$$

$$\mathrm{Re}\langle x, \sigma'_x\rangle > 0.$$

We set

$$\sigma_t = t\sigma + (1-t)\sigma' \quad \text{for} \quad 0 \leq t \leq 1.$$

σ_t also satisfies the acute angle condition and σ_t gives the homotopy between $\sigma_0 = \sigma'$ and $\sigma_1 = \sigma$. Thus $c_\sigma = c_{\sigma'}$.

Q.E.D.

<u>Definition 2.4</u> The <u>Euler obstruction</u> $Eu_0(X)$ of X at 0 is the degree of the obstraction class c_σ where $\sigma\colon \nu^{-1}(S_\varepsilon \cap X) \to \tilde{T}$ is a section satisfying the acute angle condition.

That is,

$$Eu_0(X) = \deg c_\sigma = \langle c_\sigma, \omega \rangle$$

where ω is the fundamental class of $(\nu^{-1}(B_\varepsilon \cap X), \nu^{-1}(S_\varepsilon \cap X))$.

<u>Theorem 2.5</u> (Gonzalez-Verdier formula)

$$Eu_0(X) = \deg(c_{d-1}(T-\xi) \cap [\mathcal{Y}])$$

(Cf. G. Gonzalez-Sprenberg [2])

<u>Remark 2.6</u> We consider $T-\xi$ as an element in the K-group generated vector bundles we have

$$\begin{aligned} c_{d-1}(T-\xi) &= \text{the degree } d-1 \text{ part of } c(T)/c(\xi) \\ &= \sum_{k=0}^{d-1} (-1)^k c_{d-k-1}(T) c_1(\xi)^k. \end{aligned}$$

Thus the above equality is equivalent to

$$Eu_0(X) = \sum_{k=0}^{d-1} (-1)^k \deg(c_{d-k-1}(T) c_1(\xi)^k \cap [\mathcal{Y}]).$$

The rest of this section is devoted to the proof of Theorem 2.5.

<u>Lemma 2.7</u> Let $\mathfrak{n}\colon \overline{\mathfrak{X}} \to \mathfrak{X}$ be the normalization of $\mathfrak{X}$. Let $\overline{\sigma}$ be the pull back of σ by $\tilde{e}\mathfrak{n}$. $\overline{\sigma}$ is the section of $\overline{T} = \mathfrak{n}^*T \to \overline{\mathfrak{X}}$ on $(\nu\tilde{e}\mathfrak{n})^{-1}(S_\varepsilon \cap X)$. Then, we have

$$\deg c_\sigma = \deg c_{\overline{\sigma}}.$$

<u>Proof</u>. We set $\theta = \tilde{e}\mathfrak{n}$. By definition

$$c_{\overline{\sigma}} = \theta^* c_\sigma .$$

On the other hand

$$\theta_* \Omega = \omega$$

where Ω (resp. ω) is the fundamental class of $((\nu\theta)^{-1}(B_\varepsilon \cap X), (\nu\theta)^{-1}(S_\varepsilon \cap X))$ (resp. $(\nu^{-1}(B_\varepsilon \cap X), \nu^{-1}(S_\varepsilon \cap X)).$), because θ is generically one to one.

By the projection formula, we have

$$\begin{aligned} \langle c_\sigma, \omega\rangle &= \langle c_\sigma, \theta_*\Omega\rangle \\ &= \langle \theta^* c_\sigma, \Omega\rangle \\ &= \langle c_{\overline{\sigma}}, \Omega\rangle . \end{aligned}$$

Q.E.D.

(1) Fix an hermitian metric on $\mathbb{C}^N$. We can define the orthogonal projection

$$\mathrm{proj}_L : \mathbb{C}^N \to L$$

for a linear space L. The collection

$$\xi_0 = \bigcup_{x \in \mathscr{X}} \mathrm{proj}_{T_x}(\xi_x)$$

defines a vector bundle over $\mathscr{X}$ if the representant of the germ $(X,0)$ is small enough, since by the Whitney lemma $\xi_y \subset T_y$ for every $y \in \mathcal{Y}$. We denote $\overline{\xi}_0 = \mathscr{n}^*\xi_0$, $\overline{T} = \mathscr{n}^* T$, and $\overline{\mathcal{Y}} = \mathscr{n}^{-1}\mathcal{Y}$. One has an exact sequence

$$0 \to \overline{\xi}_0 \to \overline{T} \to \overline{T}/\overline{\xi}_0 \to 0$$

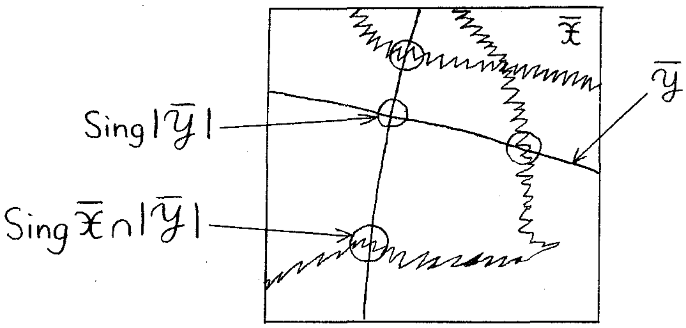
and

$$\overline{T} = \overline{\xi}_0 \oplus \overline{\xi}_0^{\perp}$$

$$\overline{\xi}_0^{\perp} \cong \overline{T}/\overline{\xi}_0.$$

(2) Consider a section σ_1 of $\overline{T}/\overline{\xi}_0$ over $|\overline{\mathcal{Y}}|$ with isolated zeros. We can build σ_1 in the way that

α) it has zeros outside $\mathrm{Sing}|\overline{\mathcal{Y}}|$ and $\mathrm{Sing}\,\mathfrak{X} \cap |\overline{\mathcal{Y}}|$.



β) σ_1 has non-degenerated zeros of index ± 1.

We can extend σ_1 as a section $\overline{\xi}_0^{\perp}$ $(\cong \overline{T}/\overline{\xi}_0)$. We denote the extension by the same letter σ_1.

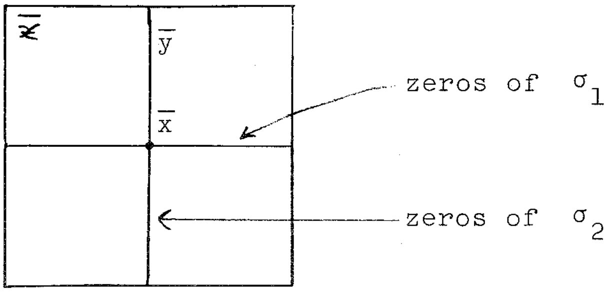
(3) Consider the section σ_2 of $\overline{\xi}_0$ obtained by pulling back the natural section $\mathrm{proj}_{T_x} \overrightarrow{0x}$. Then, $\sigma' = \sigma_2 \oplus \sigma_1$ is a section of $\overline{T}$. The zeros of $\overline{\sigma}'$ are the zeroes of $\sigma_1|\overline{\mathcal{Y}}$.

Remark 2.8 $\overline{\sigma}'$ is actually defined on a neighbourhood $\mathcal{N}$ of $\overline{\mathcal{Y}}$ and non-zero outside $\overline{\mathcal{Y}}$. If $1 \gg \varepsilon > 0$, $(\nu\tilde{e}\mathcal{N})^{-1}(B_\varepsilon \cap X) \subset \mathcal{N}$.

Lemma 2.9 The section $\overline{\sigma}'$ satisfies the acute angle condition.

Proof. Obvious.

Though the zeroes of $\sigma_1|\overline{\mathcal{Y}}$ is non-degenerated σ_2 may have multiple zeroes



Lemma 2.10 (1) The index of $\overline{\sigma}'$ at $\overline{x} \in \overline{\mathcal{Y}}$

= (the index of $\sigma_2|_{\text{zeros of } \sigma_1}$) $\times$ (the index of $\sigma_1|_{|\overline{\mathcal{Y}}|}$)

(2) (the index of $\sigma_2|_{\text{zeros of } \sigma_1}$) = the multiplicity of $\overline{\mathcal{Y}}$ at $\overline{x}$.

Proof. (1) is obvious.

(2) Let Σ_2 be the section of $\overline{\xi}$ obtained by pulling back the natural section $\overrightarrow{0x}$. We can build a homotopy connecting $(\overline{\xi}_0, \sigma_2)$ and $(\overline{\xi}, \Sigma_2)$. We have that (the index of $\sigma_2|_{\text{zeros of } \sigma_1}$)

= (the index of $\Sigma_2|_{\text{zeros of } \sigma_1}$). However, the right hand side agrees with the multiplicity of $\overline{\mathcal{Y}}$ at $\overline{x}$ by definition. Q.E.D.

Proof of Theorem 2.5 By Lemma 2.7

$$Eu_0(X) = \deg C_{\overline{\sigma}}.$$

By Lemma 2.3

$$\deg C_{\overline{\sigma}} = \deg C_{\overline{\sigma}'}.$$

Lemma 2.10 implies

$$\deg C_{\overline{\sigma}'} = \deg(c_{d-1}(\overline{T}-\overline{\xi}) \cap [\overline{\mathcal{Y}}]).$$

Since

$$c_{d-1}(\overline{T}-\overline{\xi}) = \pi^* c_{d-1}(T-\xi)$$

$$[\mathcal{Y}] = \pi_*[\overline{\mathcal{Y}}] \qquad (*)$$

one has by projection formula

$$\deg(c_{d-1}(\overline{T}-\overline{\xi}) \cap [\overline{\mathcal{Y}}]) = \deg(c_{d-1}(T-\xi) \cap [\mathcal{Y}]).$$

In conclusion we have

$$Eu_0(X) = \deg(c_{d-1}(T-\xi) \cap [\mathcal{Y}]).$$

Remark (by T. Urabe) The above equality (*) is not self-evident. However, once we establish equalities

$$[\overline{\mathcal{Y}}] = -c_1(\overline{\xi}) \cap [\overline{\mathcal{X}}]$$

$$[\mathcal{Y}] = -c_1(\xi) \cap [\mathcal{X}]$$

the equality (*) follows from the projection formula. Indeed, since $\overline{\xi} = \pi^*\xi$, and $\pi_*[\overline{\mathcal{X}}] = \mathcal{X}$, we have

$$\begin{aligned} \pi_*[\overline{\mathcal{Y}}] &= -\pi_*(\pi^* c_1(\xi) \cap [\overline{\mathcal{X}}]) \\ &= -c_1(\xi) \cap \pi_*[\overline{\mathcal{X}}] \\ &= -c_1(\xi) \cap [\mathcal{X}] \\ &= [\mathcal{Y}]. \end{aligned}$$

§3. Lê-Teissier Formula

<u>Theorem 3.1</u> (Lê-Teissier formula)

$$\mathrm{Eu}_0(X) = \sum_{k=0}^{d-1}(-1)^k m_{d-k}$$

where m_ℓ is the multiplicity of the generic ℓ-local polar variety at 0. (cf. Lê-Teissier [10].)

This theorem is an easy consequence of the next lemma.

<u>Lemma 3.2</u>

$$\deg(c_{d-1-k}(T)c_1(\xi)^k \cap [\mathcal{Y}]) = (-1)^{d-1}m_k.$$

We recall our diagram

$$\begin{tikzcd}
 & X\times G \arrow[dr, "P_2"] & \\
X\times\mathbb{P}^{N-1}\times G \supset \mathcal{Z} \arrow[r, "\tilde{e}"] \arrow[d, "\nu'"'] \arrow[dr, "\eta"] & \tilde{X} \arrow[u, hook] \arrow[r, "\gamma"] \arrow[d, "\nu"] & G=G(d,N) \\
X\times\mathbb{P}^{N-1} \supset X' \arrow[r, "e"] & X\supset X^0=X\setminus\Sigma \arrow[ur, "\gamma^0"'] &
\end{tikzcd}$$

Let $H = H_{(k+2)}$ be a linear space in $\mathbb{C}^N$ of codimension $k+2$.

$$\begin{aligned} c_{d-k-1} &= c_{d-k-1}(H) \\ &= \{T\in G\colon \dim(T\cap H_{(k+2)} \geq d-k-1\} \end{aligned}$$

is a special Schubert variety. Let $P_H\colon \mathbb{C}^N \to \mathbb{C}^{N-k-2}$ be the projection such that $\mathrm{Ker}P_H = H_{(k+2)}$. We denote the critical locus of $P_H|_{X^0}\colon X^0 \to \mathbb{C}^{N-k-2}$ by $C(P_H)$ where X^0 denotes the smooth part of X. A point $x\in X^0$ belongs to $C(P_H)$ if and only if the tangent space T_x of X at x and H do not intersect transversally. This is equivalent to saying that

$$C(P_H) = (\gamma^0)^{-1}(c_{d-k-1}).$$

By Kleiman's general point theorem (See Kleiman [7].) we can expect that for a general H, $\gamma^{-1}(c_{d-k-1})$ intersects $\nu^{-1}(\Sigma)$ properly. Thus for such H, one has

$$\overline{C(P_H)} = \nu(\gamma^{-1}(c_{d-k-1})).$$

Let $\mathbb{U} \to G = G(d,N)$ be the universal bundle. It is well-known that the dual of the Chern class of $\mathbb{U}$ is the homology class of the above mentioned Schubert cycle up to sign.

$$(-1)^{d-k-1}c_{d-k-1}(\mathbb{U}) \cap [G] = [c_{d-k-1}]$$

Now we can prove that

$$\deg c_{d-1-k}(T)c_1(\xi)^k \cap [\mathcal{Y}] \quad \text{is} \quad (-1)^{d-1}m_k$$

Notice that $\mathcal{Y}$ is a passibly non-reduced analytic space in $\{0\} \times \mathbb{P}^{N-1} \times G$. Moreover $(-1)^{d-1-k}c_{d-1-k}(T)$ corresponds to

$$(0) \times \mathbb{P}^{N-1} \times c_{d-1-k}(H_{(k+2)}) = Z_1$$

and $(-1)^k c_1(\xi)^k$ corresponds to

$$(0) \times H'_{(k)} \times G = Z_2,$$

where $H'_{(k)}$ is another generic linear subspace of $\mathbb{C}^N$ of codimension k.

<u>Lemma 3.3</u> For generic linear spaces $H_{(k+2)}$ and $H'_{(k)}$

(1) $\#(Z_1 \cap Z_2 \cap |\mathcal{Y}|) < \infty$

(2) $Z_1 \cap Z_2 \cap |\mathcal{Y}| \subset |\mathcal{Y}|_{good}$.

Here $|\mathcal{Y}|_{good}$ is some non-empty Zariski open set contained in the smooth part of $|\mathcal{Y}|$.

Proof. This is an obvious consequence of "General Pasition Theorem" by S. Kleiman. (See S. Kleiman [7].)

Corollary 3.4 For general $H_{(k+2)}$, $H'_{(k)}$.

$$\deg(c_{d-1-k}(T)c_1(\xi)^k \cap [\mathcal{Y}]) = (-1)^{d-1}\#(Z_1 \cap Z_2 \cdot \mathcal{Y}).$$

Here $\mathcal{Y}$ is counted with multiplicity.

Remark In the age of Todd, one defined characteristic classes using polar varieties. Here our viewpoint is actually a local version of his.

Lemma 3.5 We set $W = Z_1 \cap Z_2$. For $x \in W \cap |\mathcal{Y}|$ the equality

$$m_x(I_W \cdot \mathcal{O}_{\mathcal{Y},x}) = m_x(I_{\mathcal{Y}} \cdot \mathcal{O}_{\mathfrak{X} \cap W, x})$$

holds. Here $m_x(\ )$ denotes the multiplicity I_W is the ideal of W and $I_{\mathcal{Y}}$ is the ideal of $\mathcal{Y}$.

Assuming Lemma 3.5, we can verify Lemma 3.2. By definition and by Lemma 3.5

$$\#(W \cdot \mathcal{Y}) = \sum_{x \in W \cap |\mathcal{Y}|} m_x(I_W \cdot \mathcal{O}_{\mathcal{Y},x}) = \sum_x m_x(I_{\mathcal{Y}} \cdot \mathcal{O}_{\mathfrak{X} \cap W, x}).$$

We note that

$$I_{\mathcal{Y}} \cdot \mathcal{O}_{\mathfrak{X} \cap W, x} = m\,\mathcal{O}_{\mathfrak{X} \cap W, x}$$

where m is the maximal ideal of $\mathcal{O}_{X,0}$. By projection formula we have

$$m_x(I_{\mathcal{Y}} \cdot \mathcal{O}_{\mathfrak{X} \cap W, x}) = m_0(m\mathcal{O}_{\pi(\mathfrak{X} \cap W \cap U_x),0})$$

where U_x is a neighbourhood of x in $\mathbb{C}^N \times \mathbb{P}^{N-1} \times G$.

Recall that as a germ of variety

$$\bigcup_{x \in \mathcal{Y} \cap W} \eta(\mathcal{X} \cap W \cap U_x) = \Gamma_k \cap H'_{(k)}$$

where Γ_k is the k-polar variety of X. Consequently we have

$$\begin{aligned} \#(W \cdot \mathcal{Y}) &= \text{the multiplicity of } \Gamma_k \cap H'_{(k)} \text{ at } 0 \\ &= \text{the multiplicity of } \Gamma_k \text{ at } 0. \end{aligned}$$

By Corollary 3.4 we get the theorem. Q.E.D.

Now we have to verify Lemma 3.5.

<u>Proof of Lemma 3.5</u> By Kleiman's general position theorem, we can assume that

(1) $|\mathcal{Y}|$ and W is smooth at x and they intersects transversally at x.

(2) Each point of $n^{-1}(x)$ is a non-singular point of $\bar{\mathcal{X}}$, where $n: \bar{\mathcal{X}} \to \mathcal{X}$ denotes the normalization.

Let $n^{-1}(x) = \{z_1, \ldots, z_k\}$. Around z_i, we can choose a local coordinate system $t, y'_1, \ldots, y'_{\ell-1}$ such that the divisor $n^{-1}(\mathcal{Y})$ is defined by $t^{\alpha_i} = 0$ for some integer α_i. We have

$$\begin{aligned} m_x(I_W \mathcal{O}_{\mathcal{Y},x}) &= \sum_i m_{z_i}(I_W \mathcal{O}_{\bar{n}^{-1}(\mathcal{Y}), z_i}) \\ &= \sum \alpha_i \\ &= \sum_i m_{z_i}(I_{\mathcal{Y}} \mathcal{O}_{\bar{n}^{-1}(\mathcal{X} \cap W), z_i}) \\ &= m_x(I_{\mathcal{Y}} \mathcal{O}_{\mathcal{X} \cap W, x_i}). \end{aligned}$$

Q.E.D.

We now ask the following problem.

Problem We'd like to get rid of genericity condition from our Theorem 3.1.

Suppose the projections $P_1,\dots,P_d$ give local polar varieties $\Gamma(P_1),\dots,\Gamma(P_d)$ of respective dimensions $1,\dots,d$.

Find some expression e such that

$$Eu_0(X) = \sum_{k=1}^{d} (-1)^{d-k} e(\Gamma(P_k))$$

holds.

Remark Let X be a surface in $\mathbb{C}^3$. For a generic projection P_1, we have

$$Eu_0(X) = m_0(X) - m_0(\Gamma(P_1)) \qquad (*)$$

since $\Gamma(P_2) = X$ does not depend on the choice of P_2.

Suppose moreover X has an isolated singularity at 0. In this case $Eu_0(X) = 1-\mu^{(2)}$ and the equality (*) becomes

$$1 - \mu^{(2)} = m - (\mu^{(2)}+m-1)$$

where $\mu^{(2)}$ is the Milnor number of $X \cap H$ at 0, where H is a generic hyperplane passing through 0 and $m = m_0(X)$ is the multiplicity. Thus

$$m_0(\Gamma(P_1)) = \mu^{(2)}+m-1$$

is a topological character in this case.

§4. MacPherson's Theory of Singular Chern Classes

In this section we introduce readers to Chern class theory for singular varieties by MacPherson. (Cf. MacPherson [11].)

<u>Definition 4.1</u> Let V be an algebraic variety.

Let F be the smallest family of subsets of V such that

(1) A Zariski open subset belongs to $\mathcal{F}$.

(2) If $W_1, W_2, \ldots, W_k \in \mathcal{F}$, then $\bigcap_{i=1}^{k} W_i \in \mathcal{F}$.

(3) If $V, W \in \mathcal{F}$, then $V-W \in \mathcal{F}$.

A member of $\mathcal{F}$ is called a <u>constructible subset</u> of V.

<u>A locally closed subset</u> of V is an intersection of a Zariski open subset and a Zariski closed subset of V.

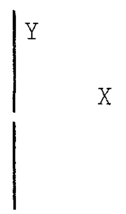
<u>Remark 4.2</u> The following two conditions are equivalent

(1) $X \subset V$ is constructible

(2) X is a finite disjoint union of locally closed subsets.

<u>Example 4.3</u> Let $V = \mathbb{C}^2$ with coordinates (x,y).

$Y = \{(x,y) \in \mathbb{C}^2 \mid x=0, y \neq 0\}$ = y-axis-origin is a locally closed subset but it is not a Zariski closed subset.



Let $X = V-Y$. X is a constructible subset of V. Obviously X is the disjoint union of $Y_1 = V-(\text{y-axis}) = V-\overline{Y}$ and the origin

<u>Definition 4.4</u> A function

$$\alpha : V \to \mathbb{Z}$$

is called <u>constructible</u> if there exist a covering $V = \coprod_{\text{finite}} Y_i$ of V by a finite family of mutually disjoint constructible subset Y_i of Y such that for each Y_i the restriction $\alpha|_{Y_i}$ is constant

<u>Remark 4.5</u> The characteristic function $\mathbb{1}_Y$ of a constructible subset Y is constructible, where $\mathbb{1}_Y$ is defined by

$$\mathbb{1}_Y(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y. \end{cases}$$

A finite sum

$$\alpha = \sum \varkappa_i \mathbb{1}_{Y_i} \qquad (*)$$

is constructible, where $\varkappa_i \in \mathbb{Z}$ and Y_i is a constructible subset.

Conversely every constructible function is expressible like (*).

<u>Lemma 4.6</u> Let V, V' and W be algebraic varieties, $f: V \to V'$ be a proper algebraic map. We suppose W is a subvariety of V. Then, the function $V' \to \mathbb{Z}$ defined by $p \mapsto \chi(f^{-1}(p) \cap W)$ is constructible, wher $\chi(\ )$ denotes the Euler Poincaie characteristic.

<u>Proof</u>. One can stratify $f|_W$. That is, there is stratification with Whitney property $W = \coprod W_k$, $V' = \coprod V'_j$ such that:

① $f|_{W_k}$ has maximal rank for every stratum W_k.

② $f(W_k) = V'_\ell$ for some ℓ.

One can see

$$\chi(f^{-1}(p) \cap W) = \sum_k \chi_c(W_k \cap f^{-1}(p)),$$

where χ_c denotes the Euler-Poincare characteristic of the cohomology group with compact support! The ritht hand side is obviously constructible.

Proposition 4.7 Let $\mathcal{V}$ be the category of compact algebraic varieties and let $\mathcal{G}$ be the category of abelian groups.

There exists a unique functor

$$\mathbb{F}: \mathcal{V} \to \mathcal{G}$$

up to natural isomorphisms such that

(1) For every compact algebraic variety $V \in \mathcal{V}$ $F(V)$ is the abelian group of constructible functions with values in $\mathbb{Z}$.

(2) For every morphism

$$f: V \to V'$$

in $\mathcal{V}$ and for every subvariety $W \subset V$.

$$\mathbb{F}(f)(\mathbb{1}_W)(p) = \chi(f^{-1}(p) \cap W)$$

for every point $p \in V'$.

Proof. If such a functor $\mathbb{F}$ exist, the uniqueness is obvious since for a morphism $f: V \to V'$ and for a constructible function $\alpha = \sum_i \varkappa_i \mathbb{1}_{W_i}$ on V,

$$\mathbb{F}(f)(\alpha) = \sum \varkappa_i F(f)(\mathbb{1}_{W_i})$$

is determined by the condition (2).

The only point to check in order to verify the existence is the equality

$$\mathbb{F}(g \circ f) = \mathbb{F}(g)\mathbb{F}(f)$$

where $f: V \to V'$ and $g: V' \to V''$ are morphisms in $\mathcal{V}$. We can see it by using suitable stratification and multiplicativeness of the Euler characteristic of compact support. Q.E.D.

<u>Theorem 4.8</u> (MacPherson) Let $\mathbb{F}: \mathcal{V} \to \mathcal{G}$ be the functor as in Proposition 4.7. Let $\mathbb{H}: \mathcal{V} \to \mathcal{G}$ be the functor of homology groups. Then, there exist a unique natural map

$$\sigma : \mathbb{F} \to \mathbb{H}$$

such that for a <u>non-singular</u> object $V \in \mathcal{V}$ the equality

$$\sigma(V)(\mathbb{1}_V) = \text{Poincare dual of the Chern class } c(V)$$

holds.

<u>Remark 4.9</u> A natural map $\sigma: \mathbb{F} \to \mathbb{H}$ is a collection of morphisms: $\sigma(V): \mathbb{F}(V) \to \mathbb{H}(V)$ of abelian groups given for each object $V \in \mathcal{V}$ such that the diagram

$$\begin{tikzcd} \mathbb{F}(V) \arrow[r, "\mathbb{F}(f)"] \arrow[d, "\sigma(V)"'] & \mathbb{F}(V') \arrow[d, "\sigma(V')"] \\ \mathbb{H}(V) \arrow[r, "\mathbb{H}(f)"] & \mathbb{H}(V') \end{tikzcd}$$

is commutative for any morphism $f: V \to V'$.

<u>Proof of uniqueness</u> Let $\alpha \in \mathbb{F}(V)$. α can be expressed as

$$\alpha = \sum n_i \mathbb{1}_{W_i}$$

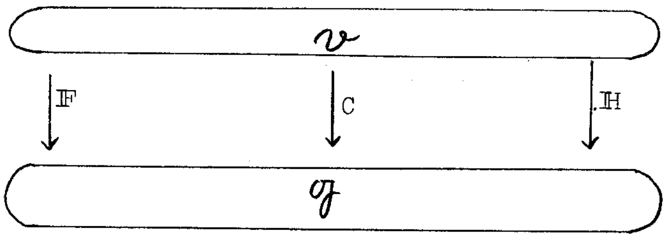
where W_i is a Zariski closed subset of V. The union $\cup W_i$ is called the support of α. The integer $d = \sup \dim W_i$ is called the dimension of the support. We proceed by induction on d. If $d = 0$, it is obvious. Assume $d > 0$. Let $W_{i_1}, \ldots, W_{i_k}$ be the components of dimension d. Let $X_i \to W_{i_j}$ be the resolution of singularities. We denote the compasition $X_j \to W_{i_j} \hookrightarrow V$ by f_j.

Let $\beta = \sum_{j=1}^{k} n_{i_j} \mathbb{F}(f_j)(\mathbb{1}_{X_j})$. We have

$$\begin{aligned}
\sigma(V)(\beta) &= \sum n_{i_j} \sigma(V)\mathbb{F}(f_j)(\mathbb{1}_{X_j}) \\
&= \sum n_{i_j} \mathbb{H}(f_j)\sigma(X_j)(\mathbb{1}_{X_j}) \\
&= \sum n_{i_j} \mathbb{H}(f_j)(\text{dual of } c(X_j)).
\end{aligned}$$

Thus $\sigma(V)(\beta)$ is uniquely determined. On the other hand by the choice of X_j's the dimension of the support of $\alpha-\beta$ is less than that of α. By induction hypothesis $\sigma(V)(\alpha-\beta)$ is uniquely determined, and we know that $\sigma(V)(\alpha) = \sigma(V)(\alpha-\beta)+\sigma(V)(\beta)$ is unique.

The proof of existence is more complicated. We don't give any complete one here. Instead we briefly sketch the MacPherson's proof. For an algebraic variety $V \in \mathcal{V}$, consider the abelian group $C(V)$ of cycles on V. An element of $C(V)$ is a finite sum $\Sigma n_i V_i$ where $n_i \in \mathbb{Z}$ and V_i is a closed subvariety of V.



$\mathbb{F}(V)$ constructible functions on V	$C(V)$ cycles on V	$\mathbb{H}(V)$ homology group of V

Consider a morphism

$$T(V): C(V) \to \mathbb{F}(V).$$

$T(V)(\Sigma \varkappa_i V_i)$ is defined as a function $x \mapsto \Sigma \varkappa_i Eu_x V_i$ (if $x \notin V_i$, we assume $Eu_x(V_i) = 0$ and for abbreviation we denote this function simply by $\Sigma \varkappa_i Eu(V_i)$ in what follows.)

Lemma 4.10

$$T(V): C(V) \to \mathbb{F}(V)$$

is an isomorphism.

Proof. We have to show that for every $\alpha = \Sigma \varkappa_i \mathbb{1}_{W_i} \in \mathbb{F}(V)$, there exists a unique cycle $\Sigma m_i V_i$ such that $T(V)(\Sigma m_i V_i) = \alpha$.

We use the induction on the dimension $d = \sup \dim W_i$ of the support of α. The case of $d = 0$ is trivial. Assume $d > 0$. Let $W_{i_1}, \ldots, W_{i_k}$ be the components of dimension d. Then

$$\alpha - T(V)\left(\sum_{j=1}^{k} \varkappa_{i_j} W_{i_j} \right) = \beta$$

has the support of dimension less than d since the function $Eu(W_i)$ has value 1 at generic points on W_i. By induction hypothesis we have a unique $\Sigma m_j V_j \in C(V)$ with $T(V)(\Sigma m_j V_j) = \beta$. The cycle $\Sigma m_j V_j + \Sigma \varkappa_{i_j} W_{i_j}$ is just what we want. Q.E.D.

Now let $\nu: \tilde{V} \to V$ be the Nash modification of V. Let $\tilde{T}$ denote the Nash bundle of V. We have the Chern class $c(\tilde{T})$ of $\tilde{T}$. The homology class $c_M(V) = \nu_*(c(\tilde{T}) \cap [\tilde{V}])$ is called the Chern-Mather class of V.

We next consider a morphism

$$c_M(V):\ C(V) \to \mathbb{H}(V)$$

defined to be

$$c_M(V)(\sum \varkappa_i V_i) = \sum \varkappa_i \ \mathrm{incl}_* \ c_M(V_i)$$

where $c_M(V_i) \in \mathbb{H}(V_i)$ is the Chern-Mather class of a subvariety V_i of V, incl: $V_i \to V$ denotes the inclusion map.

We have two morphisms

$$T(V):\ C(V) \overset{\sim}{\to} \mathbb{F}(V)$$

$$c_M(V):\ C(V) \to \mathbb{H}(V).$$

Theorem 4.11

$$\sigma(V) = c_M(V) \circ T(V)^{-1} : \mathbb{F}(V) \to \mathbb{H}(V)$$

defines the morphism σ in Theorem 4.8.

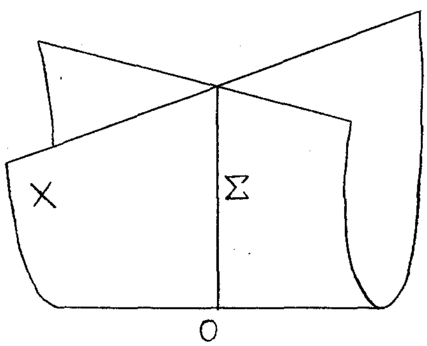
We denote $c_*(V) = \sigma(V)(\mathbb{1}_V)$ and call it the Chern-MacPherson class of V.

We give here some easy conclusions of Theorem 4.8 and Theorem 4.11.

(1) We have a unique cycle $\Sigma \varkappa_i V_i$ called the MacPherson-Schwartz cycle such that $\mathbb{1}_V = \Sigma \varkappa_i \mathrm{Eu} V_i$. Then, we have

$$c_*(V) = \sum \varkappa_i \ \mathrm{incl}_* \ c_M(V_i)$$

Remark 4.12 Let $X \subset \mathbb{C}^3$ be the Cartan's umbrella $xy^2 - z^2 = 0$



$$\begin{cases} \text{on} \quad X-\Sigma & Eu_x(X) = 1 \\ \text{on} \quad \Sigma\text{-origin} & Eu_x(X) = 2 \\ \text{at origin} \quad 0 & Eu_x(X) = 1 \end{cases}$$

The macPherson-Schwartz cycle is $X-\Sigma+0$. Here the value of $Eu_x(X)$ is computed by the Lê-Teissier formula.

(2) Consider the map $f:V \to *$, where $*$ denotes the one point. We have the commutative diagram

$$\begin{tikzcd} \mathbb{F}(V) \arrow[r, "\sigma(V)"] \arrow[d, "\mathbb{F}(f)"'] & \mathbb{H}(V) \arrow[d, "\mathbb{H}(f)"] \\ \mathbb{F}(*) \arrow[r, "\sigma(*)"] & \mathbb{H}(*) \cong \mathbb{Z} \end{tikzcd}$$

By definition of $\mathbb{F}(f)$

$$\mathbb{F}(f)(\mathbb{1}_V) = \chi(V).$$

And

$$\begin{aligned} \deg c_*(V) &= \mathbb{H}(f)(c_*(V)) \\ &= \mathbb{H}(f)\circ\sigma(V)(\mathbb{1}_V) \\ &= \sigma(*)\circ\mathbb{F}(f)(\mathbb{1}_V) \\ &= \chi(V). \end{aligned}$$

Now we get the Gauss-Bonnet property

$$\chi(V) = \deg c_*(V).$$

Remark 4.12 What is left to verify for us now is the following Ⓐ. The other parts are easy to check.

Ⓐ Let $f: V \to W$ be a map of compact algebraic varieties. Then, the diagram

$$\begin{tikzcd} \mathbb{F}(V) \arrow[r] \arrow[d, "\sigma(V)"'] & \mathbb{F}(W) \arrow[d, "\sigma(W)"] \\ \mathbb{H}(V) \arrow[r] & \mathbb{H}(W) \end{tikzcd}$$

is commutative for the morphisms $\sigma(V)$, $\sigma(W)$ defined in Theorem 4.11.

Reduction. It is enough to prove next Ⓑ

Ⓑ For any map $g: X \to V$ such that X is non-singular, the equality

$$\sigma(V)\mathbb{F}(g)(\mathbb{1}_X) = \mathbb{H}(g)\sigma(X)(\mathbb{1}_X)$$

holds.

Let $\alpha \in \mathbb{F}(V)$. We have integers $k_i \in \mathbb{Z}$ and morphisms $g_i: X_i \to V$ with non-singular X_i such that

$$\alpha = \sum k_i \mathbb{F}(g_i)(\mathbb{1}_{X_i}).$$

Assume Ⓑ. We have for $f: V \to W$

$$
\begin{aligned}
\sigma(W)\cdot\mathbb{F}(f)(\alpha) &= \sum k_i\sigma(W)\mathbb{F}(f\circ g_i)(\mathbb{1}_{X_i}) \\
&= \sum k_i\mathbb{H}(f)\mathbb{H}(g_i)\sigma(X_i)(\mathbb{1}_{X_i}) \\
&= \mathbb{H}(f)(\sum k_i\mathbb{H}(g_i)\sigma(X_i)(\mathbb{1}_{X_i}) \\
&= \mathbb{H}(f)(\sum k_i\sigma(V)\mathbb{F}(g_i)(\mathbb{1}_{X_i})) \\
&= \mathbb{H}(f)\sigma(V)(\sum k_i\mathbb{F}(g_i)(\mathbb{1}_{X_i})) \\
&= \mathbb{H}(f)\sigma(V)(\alpha)
\end{aligned}
$$

Thus we can conclude (A).

(B) is equivalent to the next (C)

(C) Let $g: X \to V$ be a map with non-singular X. Then, there exists a cycle $\sum n_i V_i$ on V such that

(1) $\sum n_i \mathrm{Eu} V_i = \mathbb{F}(g)(\mathbb{1}_X)$

(2) $\sum \varkappa_i \; \mathrm{incl}_* \; c_M(V_i)(V_i) = \mathbb{H}(g)(\text{Dual of } \; c_*(X))$.

Here $\mathbb{F}(g)(\mathbb{1}_X)$ is the function on V defined by $x \mapsto \chi(g^{-1}(x))$.

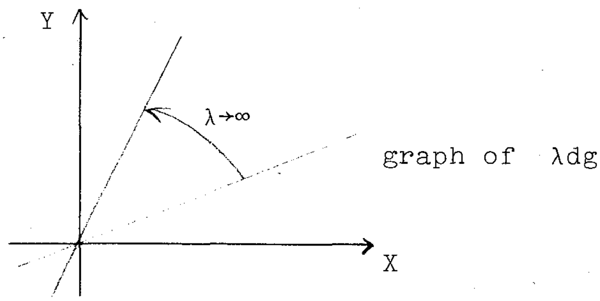
The situation is very similar to where we treat polar varieties.

$$
\begin{tikzcd}
T_i \arrow[r] & \tilde{V}_i \arrow[d, "\nu_i"] \arrow[dr] & \\
& V_i & G
\end{tikzcd}
$$

MacPherson is not doing like that. He picks up a certain cycle satisfying the above condition (1), (2). Each line of his proof is not difficult to understand. As we have not a better understanding at this moment, we advice the reader to consult

MacPherson's paper ([11]). We sketch briefly MacPherson's "Graph Construction".

Assume V is embedded in a non-singular variety Y. The compasition $X \xrightarrow{g} V \hookrightarrow Y$ is also denoted by g. Let $d = \dim X$. We denote by $G_d(TX \oplus g^*TY) \xrightarrow{\pi} X$ the bundle of Grassmann varieties of d-planes in the vector bundle $TX \oplus TY$. Each fibre $\pi^{-1}(x) = G_d(TX \oplus TY)_x$ is the Grassmann varieties of d-planes in $(TX \oplus g^*TY)_x$. For each $\lambda \in \mathbb{C}$, a section $s_\lambda: X \to G_d(TX \oplus g^*TY)$ is defined to be $s_\lambda(x)$ = the graph of the map $\lambda dg_x: T_xX \to T_{g(x)}Y$, which is considered as a d-linear space in $T_xX \oplus T_{g(x)}Y$. We get a map $\varphi: X \times \mathbb{C} \to G_d(TX \oplus g^*TY) \times \mathbb{P}^1$ with $\varphi(x,\lambda) = (s_\lambda(x),\lambda)$. Let $W = \overline{\mathrm{Im}\varphi}$ and $Z_\infty = \Sigma m_i V_i' = W \cap (G_d \times \{\infty\})$, where V_i' is irreducible and m_i is the multiplicity as a divisor. Then, $V_i = g\pi(V_i')$'s constitute the components of the cycle in Ⓒ. The coeffcient $n_i = p_i m_i$ is the multiple of m_i by a certain number p_i.



Chapter III. Whitney Stratification

§1. Whitney Stratification

We have the next problem.

<u>Problem</u>. Find the cycle $\sum n_i V_i$ on V such that $\mathbb{1}_V = \sum n_i \mathrm{Eu} V_i$.

Denote

$$e_x(X) = (m_1, \ldots, m_d),$$

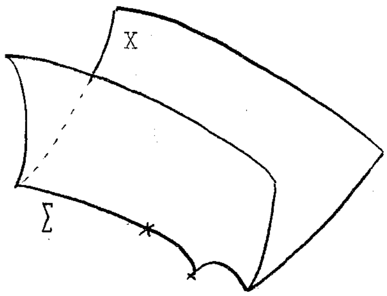
where m_i be the multiplicity of the generic polar variety of dimension i of X at x. $x \in X$ is a non-singular point if and only if $e_x(X) = (0,\ldots,0,1)$. Note that there also exists a point with $e_x(X) = (0,\ldots,0,k)$ $(k > 1)$. For example the swallow tail has $e_x(X) = (0,3)$. Recall that if $m_j = 0$ for $1 \leq j \leq r$, then the dimension of the set $\nu^{-1}(x)$ of limit tangents is less than or equal to $d-1-r$. (Proposition 4.8 in Chapter I.) Let $V = F_0 = F_{00}$. We put

$$\begin{aligned} F_1 &= \Sigma \text{ (the singular locus)} \\ &= \{C_x(X) \neq (0,\ldots,0,1)\}. \end{aligned}$$

Let $F_1 = \bigcup_{j \in J_1} F_{1j}$ be the decomposition into irreducible components and let

$$F'_{1j} = \{x \in F_{1j} \text{ where } e_x(V) \text{ does not have the generic value along } F_{1j}\}.$$

$\underline{\text{Example}}$ Let $V \subset \mathbb{P}^3$ be a surface.



$F_1 = \sum = \bigcup_{j \in J_1} F_{1j}$ is a union of curves.

F'_{1j} is a finite number of "had" points.

Next put

$$F_2 = (\bigcup_{j \in J_1} F'_{1j}) \cup \textstyle\sum(F_1) = \bigcup_{j \in J_2} F_{2j},$$

where $\sum(F_1)$ is the singular locus of F_1, and F_{2j}'s are irreducible components. In particular $F_1 \setminus F_2$ is non singular. Put

$F'_{2j} = \{x \in F_{2j};$ where $e_x(V)$, $e_x(F_{1k})$ $(k \in J_1)$ do not have the generic value along $F_{2j}\}$.

($\underline{\text{Convention}}$ $e_x(F_{1k}) = (0,\ldots,0)$ if $x \notin F_{1k}$.)

$$F_3 = (\bigcup_{j \in J_2} F'_{2j}) \cup \textstyle\sum(F_2).$$

We repeat this procedure.

If $F_r = \bigcup_{j \in J_r} F_{rj}$ (F_{rj} is irreducible.), put

$F'_{rj} = \{x \in F_{rj}$ where $e_x(F_{sj})$ $(s < r,\ j \in J_s)$

do not have the generic values along $F_{rj}\}$

and

$$F_{r+1} = (\bigcup_{j \in J_r} F'_{rj}) \cup \textstyle\sum(F_r).$$

Remark 1.1 1. Obviously this procedure ends in finite steps.

2. $S_{ij} = F_{ij} \setminus \bigcup_{\ell > i} F_{\ell k} = F_{ij} \setminus F_{i+1}$ is smooth and by definition $Eu_x(F_{rs})$ is constant on S_{ij}. We denote $Eu_{ij}(r,s) = Eu_x(F_{rs})$ for $x \in S_{ij}$.

3. We can determine the coefficient $\varkappa_{ij}$ step by step as follows and we can compute the MacPherson-Schwartz cycle $\sum_i \sum_{j \in J_i} \varkappa_{ij} S_{ij}$ explicitly.

① $\varkappa_{00} = 1$

② $\varkappa_{ij} + \sum_{\substack{r<i \\ s \in J_r}} \varkappa_{rs} E_{ij}(r,s) = 1.$

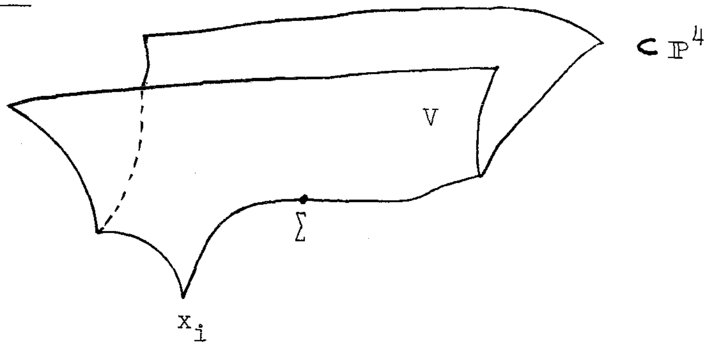
Let $x \in S_{ij} \subset V$. Assume the germ (V,x) is embedded in $(\mathbb{C}^N, x)$. Let B_ε is a ball in $\mathbb{C}^N$ with center x and with radius ε. Let L be a general linear space of codimension $\dim S_{ij}+1$ near enough to x however, which is not passing through x.

We denote $\chi_{ij} = \chi(L \cap V \cap B_\varepsilon)$.

We can prove the next theorem . We do not give the proof here.

Theorem 1.2. (1) χ_{ij} is independent of $x \in S_{ij}$.
(2) $\varkappa_{ij} = 1-\chi_{ij}$, where $\varkappa_{ij}$ is as in Remark 1.1, 3.

Example



Let V be a hypersurface in $\mathbb{P}^4$, $(\dim V = 3)$. Assume the singular locus $\Sigma = \bigcup_i \Sigma_i$ has dimension 1.

Let x_j's be singular points of Σ. Let y_k's be the smooth points of Σ where $e_y(X)$ or $e_y(\Sigma)$ do not have the generic value along a Σ_i.

Then, the MacPherson-Schwartz cycle has the following form

$$V^0 + \sum_i \mu_i \Sigma_i^0 + \sum_j \varkappa_j x_j \qquad (*)$$

where $V^0 = V \setminus \Sigma$

$$\Sigma_i^0 = \Sigma \setminus \{x_j\text{'s},\ y_k\text{'s}\}.$$

Let L be a general linear 3-space near enough a y_k. Then $L \cap V \cap B_\varepsilon$ has only an isolated singularity. Thus it is contractible. We have $\chi(L \cap V \cap B_\varepsilon) = 1$ and the coefficients of y_k's in $(*)$ are zero.

Let H be a general 2-linear space passing through a general point on Σ_i^0. It follows that $\mu_i = \mu(X \cap H \cap B_\varepsilon)$, the Milnor number of $X \cap H$ cut out by a small ball B_ε with center x.

Next theorem implies that our construction is "canonical".

Theorem 1.3 (B. Teissier) $V = \amalg S_{ij}$ is the coarsest stratification of V such that

(1) $\Sigma,\ \Sigma_1 = \Sigma(\Sigma),\ \Sigma_2 = \Sigma(\Sigma_1), \ldots,$

are union of strata.

(2) One has the Whitney condition.

The proof of this theorem will be given in §3.

§2. Chern-Mather Class of Projective Varieties

Let V be an projective variety. Let $\sum n_i V_i$ be the MacPherson-Schwartz cycle of V. We have

$$c_*(V) = \sum n_i \operatorname{incl}_* c_M(V_i).$$

We can not say that the right-hand side has an explicit meaning. What is the Chern-Mather class $c_M(V_i)$ of V_i?

$c_M(V_i)$ is very complicated and incl_* is not easy to compute. The part of degree 0

$$c_*(V)_0 = \chi(V) = \sum n_i c_M(V_i)_0$$

is rather easy. In this section we give a geometrical meaning to $c_M(V_i)_0$.

Let $X \subset \mathbb{P}^N$ be a projective algebraic variety of dimension $d-1$. We would like to define the "global" Nash modification and "global" polar varieties. "Global" polar varieties are called <u>polar cycles</u>.

<u>Nash modification</u> Let X^0 be the smooth part of X. We can consider two different Gauss-map.

(1) We have the bundle of Grassmann varieties $G_{d-1}(T(\mathbb{P}^N)) \to \mathbb{P}^N$. A section $\gamma^0: X^0 \to G_{d-1}(T(\mathbb{P}^N))$ is defined to be $\gamma^0(x) = T_x(X) \subset T_x(\mathbb{P}^N)$. Set $\tilde{X}$ = the closure of $\gamma^0(X^0)$ in $G_{d-1}(T(\mathbb{P}^N))$. We call a map $\nu: \tilde{X} \to X$ induced by the projection $G_{d-1}(T(\mathbb{P}^N)) \to \mathbb{P}^N$, <u>the Nash modification</u> and the restriction $\tilde{T} \to \tilde{X}$ of the universal bundle $\mathbb{U} \to G_{d-1}(T(\mathbb{P}^N))$ to $\tilde{X}$ is <u>the Nash bundle</u> of X.

For a hyperplane $H \subset \mathbb{P}^N$, $X_H = X \setminus H$ is a subvariety of $\mathbb{P}^N \setminus H \cong \mathbb{C}^N$. We know that $\nu: \tilde{X}_H = \nu^{-1}(X_H) \to X_H$ agrees with the

Nash modification and that $\tilde{T}|_{\tilde{X}_H}$ is the Nash bundle discussed in preceding sections.

(2) For any smooth point $x \in X^0$ we have the realized tangent space L_x. L_x is the union of 1-dimensional projective subspaces of $\mathbb{P}^N$ which have multiplicity greater than 1 at x. L_x is a linear subspace of $\mathbb{P}^N$ and it is isomorphic to the projective space $\mathbb{P}^{d-1}$, (not the affine space $\mathbb{C}^{d-1}$.)

A map

$$\beta : X^0 \to G_d(\mathbb{C}^{N+1})$$

is defined to be $\beta(x)$ = the cone over $L_x \subset \mathbb{C}^{N+1}$. Sometimes β is called the Gauss map for X.

Let $\tilde{\tilde{X}}$ be the closure of the graph of β in $X \times G_d(\mathbb{C}^{N+1})$. We can see that $\tilde{\tilde{X}}$ is isomorphic to the Nash modification $\tilde{X}$.

<u>Polar cycles</u> Let $\hat{X} \subset \mathbb{C}^{N+1}$ be the cone over X. Local polar varieties of $(\hat{X},0)$ are cones. Thus we have projective subvarieties of X associated to those cones. We call them <u>polar cycles</u> of X and the one of dimension k is denoted M_k and called k-polar cycle.

Let $L \subset \mathbb{P}^N$ be a general linear subspace with $\dim L = N-k-2$, $p_L : \mathbb{P}^N - L \to \mathbb{P}^{1+k}$ be the projection with center L. Then, M_k agrees with the colsure of the critical locus of the map $X-L \to \mathbb{P}^{1+k}$ induced by p_L.

<u>Remark</u> It follows that for any point $x \in X_H = X \setminus H$, $M_1 \setminus H$, $M_2 \setminus H, \cdots, M_d \setminus H$ are the local polar varieties of (X_H, x).

J. A. Todd showed that for a non-singular projective variety

$X \subset \mathbb{P}^N$.

$$\text{Dual of } \quad c_{d-1}(TX) = \sum_{i=0}^{d-1} (-1)^{d-1-i}(i+1)c_1(L)^i \cap [M_i],$$

where L is the universal line bundle of $\mathbb{P}^N$.

R. Piene extended this result and obtained

$$c_M(X)_0 = \sum_{i=0}^{d-1} (-1)^{d-1-i}(i+1)c_1(L)^i \cap [M_i] ,$$

for any $X \subset \mathbb{P}^N$, which is passibly singular.

By this formula we get the next proposition.

<u>Proposition 2.1</u> Let $X \subset \mathbb{P}^N$ be a projective variety of dimension d-1. Let $\hat{X}$ denote the affine cone of X. Let

$$0 \in D_{d-1} \subset \cdots \subset D_1 \subset D_0 = \mathbb{C}^{N+1}$$

be a general flag with codim $D_k = k$. Then we have

$$c_M(X)_0 = \sum_{k=0}^{d-1} Eu_0(\hat{X} \cap D_k).$$

<u>Proof</u> $c_1(L)^i \cap [M_i]$ agrees with the multiplicity of the cone and the polar variety Γ_j' of $X_1 \cap D_k$ is equivalent to $\Gamma_{j+k} \cap D_k$ where Γ_{j+k} is the polar variety of $\hat{X}$. Thus the Lê-Teissier formula and the Piene's formula imply the above one.

§3. Numerical Characterization of Whitney Condition

We proceed to the proof of Theorem 1.3, which is obviously the direct consequence of the next theorem, (B. Teissier [15]). However, we can point out some incomplete points in his proof.

<u>Theorem 3.1</u> Let $(X,0) \subset (\mathbb{C}^N,0)$ be a reduced germ of

analytic spaces in $(\mathbb{C}^N,0)$ of pure dimension d. Let $(Y,0) \subset (X,0)$ be a <u>non-singular</u> sub germ of X. Let X^0 denote the non-singular part of X. We get the following diagram.

$$\begin{tikzcd} \mathcal{X}_Y \arrow[r, "\tilde{e}_Y"] \arrow[d, "\nu_Y"'] \arrow[dr, "\eta_Y"] & \tilde{X} \subset X \times G \arrow[d, "\nu"] \\ X_Y \arrow[r, "e_Y"] & X \end{tikzcd}$$

ν : the Nash modification

e_Y: the blowing up of Y

$\tilde{e}_Y$: the blowing up of $\nu^{-1}(Y)$

ν_Y: induced map by universality of the blowing-up

$\eta_Y = e_Y \cdot \nu_Y = \nu \cdot \tilde{e}_Y$

Then, the following assertions are equivalent.

(i) For any $y \in Y$

$$m_y(\Gamma_k(X,y)) = m_0(\Gamma_k(X,0))$$

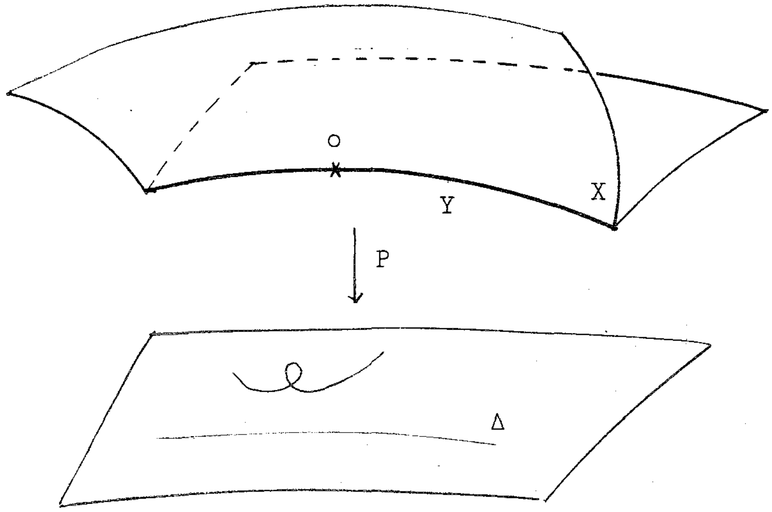
for any integer k with $1 \leq k \leq d$.

(ii) $\eta_Y|_Y$ is an equidimensional map. (i.e. all fibres have the same dimension.)

(iii) X^0 along Y satisfies the Whitney condition at every point $y \in Y$.

<u>Example 3.2</u> Let $X \subset \mathbb{C}^3$ be a surface, $Y = \sum(X)$ be the singular locus. We assume that Y is smooth and the origin o

is on Y.



Let $P : X \to \mathbb{C}^2$ be a generic projection. Assume that the condition (i) in Theorem 3.1 is satisfied. We will see that the condition (iii) is also satisfied.

Anyway, let L_y denote a general line in $\mathbb{C}^2$ passing through $P(y)$ for $y \in \Sigma$. We have an equality

$$\mu(P^{-1}(L_y), y) + \deg_y P - 1 = m_{p(y)}(\Delta)$$

where Δ is the discriminant of P. Under the assumption of (i) $\deg_y P$ is independent of $y \in Y$ and at every point $y \in Y$ the polar curve is void. It follows that $P(Y)$ is the only component of the discriminant and that $m_{p(y)}(\Delta)$, $y \in Y$ is constant. The above equality implies $\mu(P^{-1}(L_y), y)$ is constant for $y \in Y$. By Proposition 3.5 in Chapter I, we see that the condition (iii) holds:

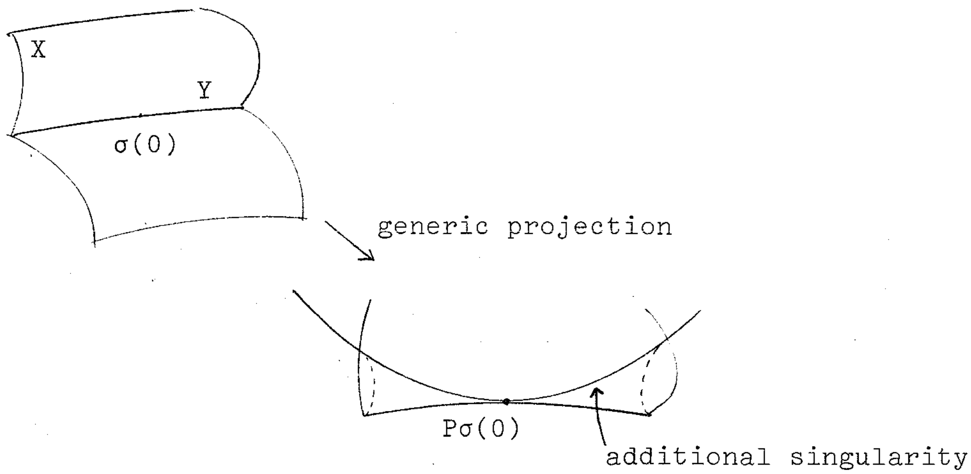
Remark 3.3 We have a surface X in $\mathbb{C}^4$ such that

(0) The projection $\mathbb{C}^4 = \mathbb{C}^3 \times \mathbb{C} \to \mathbb{C}$ defines a map $X \xrightarrow{\pi} \mathbb{D} = \{|t|<1\}$ and $\pi^{-1}(t)$ is a curve for every $t \in \mathbb{D}$.

(1) There exist a section $\sigma : \mathbb{D} \to X$ and the singular locus of X is $\sigma(\mathbb{D})$.

(2) The pair $(X-\sigma(\mathbb{D}),\sigma(\mathbb{D}))$ satisfies the Whitney condition.

(3) Let $P : \mathbb{C}^4 \to \mathbb{C}^3$ be a generic projection $Z = P(X)$ be the image. Z has an aditional singular locus passing through $P(\sigma(0))$.



Thus for general surfaces $X \subset \mathbb{C}^N$, the Whitney condition does not imply the constancy of the multiplicity of the discriminant of a generic projection $X \to \mathbb{C}^2$. (The constancy of this multiplicity is called the equisingularity condition in the sense of Zariski.)

However for surfaces in $\mathbb{C}^3$, the Whitney condition and the Zariski's equisingularity condition are equivalent. (Cf. Lê-

Teissier [9].)

Proof of Theorem 3.1. (i) $\Rightarrow$ (ii)

Step 1 (ν is equidimensional.)

Let $\dim Y = t$, $\dim \nu^{-1}(Y) = d'$. For a general $y \in Y$, $\dim \nu^{-1}(y) \leq d'-t$. By Proposition 4.8 in Chapter I, $\Gamma_k(X, y) = \emptyset$ for $1 \leq k \leq \ell = d-d'+t$. By the assumption (i) $\Gamma_k(X,0) = \emptyset$ for $1 \leq k \leq \ell$. Again by Proposition 4.8 in Chapter I, we have $\dim \nu^{-1}(0) \leq d'-t$ and since the dimension of the fibre is upper-semi-continuous, $\dim \nu^{-1}(0) = d'-t$.

Step 2 (η_Y is also equidimensional.)

We use induction on $d-t = \operatorname{codim}_X Y$. If $d-t = 1$, assertion (i) implies the equi-multiplicity of X along Y, and one sees that e_Y is also finite. Thus, η_Y is finite.

Suppose $d-t > 1$. Let $\mathcal{Y}_0$ be an irreducible component of $\eta_Y^{-1}(0) \subset \mathbb{P}^{N-t} \times G$, and $\mathcal{V}_0$ be the projection of $\mathcal{Y}_0$ to $\mathbb{P}^{N-t}$.

If $\dim \mathcal{V}_0 \geq 1$, one can find a smooth hypersurface H such that

(1) H containes Y.

(2) The induced map $(X \cap H)^\wedge \to \widetilde{X \cap H}$ from the strict transform of $X \cap H$ by ν to the Nash modification of $X \cap H$, is finite.

(3) The strict transform $(X \cap H)'^\wedge$ of $X \cap H$ by η_Y intersects $\mathcal{Y}_0$.

By the induction hypothesis and by the finiteness of the above map, we have

$$\dim((X \cap H)'^\wedge \cap \mathcal{Y}_0) \leq d'-t-1$$

thus, we have $\dim \mathcal{Y}_0 \leq d'-t$.

If $\dim \mathcal{U}_0 = 0$, we still have $\dim \mathcal{Y}_0 \leq d'-t$ because $\mathcal{Y}_0 \subseteq \mathcal{V}_0 \times \mathcal{W}_0$ with $\mathcal{W}_0 \subset \nu^{-1}(0)$ and $\dim \nu^{-1}(0) = d'-t$.

(ii) $\Rightarrow$ (iii)

STEP 1 Let $x_n \in X^0$ and $y_n \in Y$ be convergent sequences such that

① $z = \lim x_n = \lim y_n \in Y$

② the limit $T = \lim T_{x_n} X$ exists

③ the limit $\ell = \lim \overline{y_n x_n}$ exists.

We would like to show that $\ell \subset T$.

Let $P : (\mathbb{C}^N, 0) \to (Y, 0)$ be a projection. One has

$$\overrightarrow{y_n x_n} = \overrightarrow{y_n P(x_n)} + \overrightarrow{P(x_n) x_n}$$

By choosing a subsequence x_{n_k} of x_n if necessary, we can assume that the limit of secants $\ell_1 = \lim \overline{y_n P(x_n)}$ and $\ell_2 = \lim \overline{P(x_n) x_n}$ exist. By definition $\ell_1 \subset T_z Y \subset T$. Thus it is sufficient to see that $\ell_2 \subset T$. Consequently we can assume that $y_n = P(x_n)$.

STEP 2 For $x \in X^0$, one can define the angle function $\beta(x)$ between $T_x X$ and $\overline{xP(x)}$. Namely,

$$\beta(x) = \sup_{\substack{v \in \overline{xP(x)} - (0) \\ w \in T_x X - (0)}} \frac{|\langle v, w\rangle|}{\|v\|\|w\|},$$

where $\langle\ ,\ \rangle$ denotes a Hermitian form on $\mathbb{C}^N$. It is easy to see that we have an extension $\tilde{\beta} : \mathcal{X}_Y \to \mathbb{R}$ such that $\beta \circ \eta_Y(p) = \tilde{\beta}(p)$ for $p \in \eta_Y^{-1}(X^0)$. The Whitney condition at 0 is equivalent to that $\tilde{\beta}|_{\eta_Y^{-1}(0)} \equiv 1$. However, for a non-empty Zariski open set

$U \subset Y$, and for $y \in U$, $\beta|_{\eta_Y^{-1}(y)} \equiv 1$ (Cf. H. Hironaka [5],[6]). The assumption (ii) implies that $\tilde{\beta}^{-1}(U)$ is dense in $\tilde{\beta}^{-1}(Y)$. Thus we have $\tilde{\beta}|_{\eta_Y^{-1}(0)} \equiv 1$.

(iii) $\Rightarrow$ (i)

STEP 1 (We can assume $\dim Y = 1$.)

For any $y \in Y$ which is sufficiently near to 0, we can choose a smooth curve Y_1 passing through both 0 and y. The Whitney condition for the pair (X^0, Y) implies the Whitney condition for (X^0, Y_1). Thus, if we can prove the theorem under an additional condition that $\dim Y = 1$, we conclude that $m_y(\Gamma_k(X, y)) = m_0(\Gamma_k(X, 0))$ for any k.

STEP 2 (Whitney condition $\Rightarrow$ $\dim \nu^{-1}(0) \leq d-2$)

Assume $\dim Y = 1$. Let $P\colon \mathbb{C}^N \to \mathbb{C}^{1+d}$ be a generic projection, $X_1 = P(X)$. Teissier used the assertion that the Whitney condition for (X, Y) implies the Whitney condition for $(X_1, P(Y))$, without proof. However this assertion is not obvious. This is one of the incomplete parts of his proof.

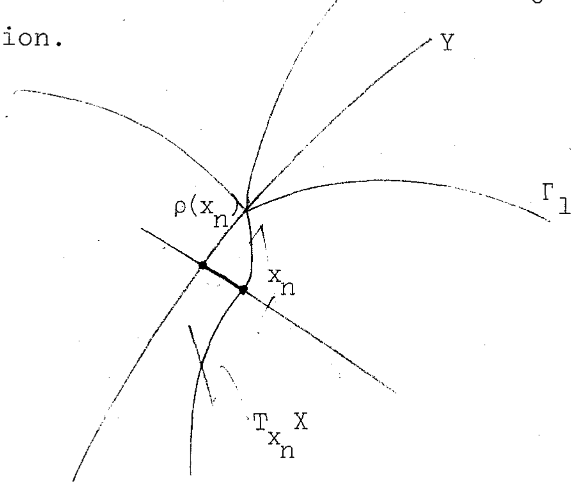
So, here we only show that assuming the next lemma, we can accomplish the proof.

Lemma 3.4 For a general projection $Q : \mathbb{C}^N \to \mathbb{C}^2$, the union $\Gamma_1 \cup Y$ of the polar curve Γ_1 associated to Q and Y has the following property; each limit direction at 0 of the secant of $\Gamma_1 \cup Y$ does not contained in $\operatorname{Ker} Q$.

By Proposition 4.8 in Chapter I, we know that $\dim \nu^{-1}(0) \leq d-2$ is equivalent to that $\Gamma_1 = \emptyset$.

Assume $\Gamma_1 \neq \emptyset$ for any generic projection $Q : \mathbb{C}^N \to \mathbb{C}^2$. We fix a retraction $\rho : \mathbb{C}^N \to Y$. Let $x_n \in \Gamma_1 \setminus (0)$ be a sequence which tends to 0. Consider the secants $\overline{\rho(x_n)x_n}$, there limit ℓ

of them and the limit T of $T_{x_n}X$. By the Whitney condition $T \supset T_0 Y$ and by the construction $\dim (T \cap \mathrm{Ker}\, Q) \geq \dim T - 1$. We can assume, $T_0 Y \not\subset \mathrm{Ker}\, Q$. Thus we have $T = T \cap \mathrm{Ker}\, Q + T_0 Y$. On the other hand, by the Whitney condition $T \supset \ell$ and by lemma 3.4, $\ell \not\subset \mathrm{Ker}\, Q$. We have $T = \mathrm{Ker}\, Q \cap T + \ell$. However $\mathbb{C}^N = \ell + T_0 Y + \mathrm{Ker}\, Q$, which is a contradiction.



STEP 3 Since $\dim \nu^{-1}(0) \leq d-2$, a sufficiently general non-singular hypersurface $H \supset Y$ is transversal to every limit tangent space of X. Thus we can conclude that $(X \cap H, Y)$ satisfies the Whitney condition. By induction on d, we get the theorem thanks to the next lemma.

Lemma 3.5. Suppose $\dim Y = 1$ and $m_0(\Gamma_1(X,0)) = 0$. Then, for any general smooth hypersurface H containing Y, we have

(1) $m_0(\Gamma_{k+1}(X,0)) = m_0(\Gamma_k(X \cap H, 0))$ for $k \geq 2$.

(2) If $\Gamma_1(X \cap H, 0) = \emptyset$, then $\{\Gamma_2(X,y)\}_{y \in Y}$ is equimultiple along Y.

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